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where the 'degree' d corresponds to the cubic form. Such a 4-tuple is admissible iff $d(2x + W)^3 \equiv (p + 24T) \cdot (2x + W) \pmod{48}$ holds for every integer x . This is equivalent to $p \equiv 4d \pmod{24}$ if $\bar{W} = 0$, and to $p \equiv d + 24T \pmod{48}$ with $d \equiv 0 \pmod{2}$ if $\bar{W} \neq 0$.

Two admissible 4-tuples (\bar{W}, \bar{T}, d, p) and $(\bar{W}', \bar{T}', d', p')$ are equivalent iff $\bar{W}' = \bar{W}$, $\bar{T}' = \bar{T}$ and $(d', p') = \pm(d, p)$. Taking the degree d non-negative, we find:

PROPOSITION 1. *There is a 1-1 correspondence between oriented homeomorphism types of cores X_0 with $b_2(X_0) = 1$, and 4-tuples (\bar{W}, \bar{T}, d, p) , normalized so that $d \geq 0$, and $p \geq 0$ if $d = 0$, which satisfy $p \equiv 4d \pmod{24}$ if $\bar{W} = 0$, and $d \equiv 0 \pmod{2}$, $p \equiv d + 24T \pmod{48}$ if $\bar{W} \neq 0$.*

In order to classify the associated homotopy types we first have to determine the subgroup U_F associated to a given cubic form F . By definition we find $U_F = 0$ if $d \equiv 0 \pmod{2}$, $U_F = \mathbf{Z}/2$ if $d \equiv 1 \pmod{2}$. Two normalized 4-tuples (\bar{W}, \bar{T}, d, p) and $(\bar{W}', \bar{T}', d', p')$ are weakly equivalent iff $d' = d$, $\bar{W}' = \bar{W}$, and $p + 24T \equiv p' + 24T' \pmod{48}$ if $d \equiv 0 \pmod{2}$, $p \equiv p' \pmod{24}$ if $d \equiv 1 \pmod{2}$.

Putting everything together, we find a single oriented homotopy type for every odd degree $d \geq 0$, which is necessarily spin, and 3 oriented homotopy types for every even degree $d \geq 0$; one of these 3 types has $\bar{W} \neq 0$, the other two are spin, and they are distinguished by $p + 24T \pmod{48}$ i.e. $p \equiv 4d \pmod{48}$, or $p \equiv 4d + 24 \pmod{48}$.

2. REALIZATION OF CUBIC FORMS

In the previous section the (homotopy) topological classification of 1-connected, closed, oriented, 6-dimensional manifolds with torsion-free homology has been transformed into an arithmetical moduli problem: to describe the sets of (weak) equivalence classes of admissible systems of invariants. In this section we begin to investigate the latter problem; we give a simple criterion for the realizability of cubic forms by smooth manifolds, and we describe, at least in principle, the classification of homotopy types of manifolds with a given cohomology ring.

2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS

Let (r, H, w, τ, F, p) be a system of invariants as in section 1; recall that it is admissible iff for every $W \in H$, $T \in H^\vee$ with $\bar{W} = w(\text{mod } 2)$, $\bar{T} \equiv \tau(\text{mod } 2)$ the following congruence holds:

$$(*) \quad W^3 \equiv (p + 24T)(W) \pmod{48}.$$

LEMMA 1. (r, H, w, τ, F, p) is admissible if and only if there exist $W_\circ \in H$, $T_\circ \in H^\vee$ with $\bar{W}_\circ \equiv w(\text{mod } 2)$, $\bar{T}_\circ \equiv \tau(\text{mod } 2)$, such that

- i) $W_\circ^3 \equiv (p + 24T_\circ)(W_\circ) \pmod{48}$
- ii) $p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2 \pmod{24} \quad \forall x \in H$.

Proof. Obvious since the set of integral lifts of w is a coset $W_\circ + 2H$.

DEFINITION 3. Let $F \in S^3H^\vee$ be a symmetric trilinear form on a finitely generated free abelian group H . An element $W \in H$ is characteristic for F iff

$$(**) \quad x \cdot y \cdot (x + y + W) \equiv 0 \pmod{2} \quad \forall x, y \in H.$$

LEMMA 2. $W \in H$ is a characteristic element for $F \in S^3H^\vee$ if and only if the function $l_W: H \rightarrow \mathbf{Z}$, $l_W(x) := 4x^3 + 6x^2W + 3xW^2$ is linear in x modulo 24.

Proof. $l_W(x + y) = l_W(x) + l_W(y) + 12(x^2y + xy^2 + xyW)$, whence the assertion.

The existence of characteristic elements is a necessary and sufficient condition for a cubic form $F \in S^3H^\vee$ to be realizable by a manifold. In fact, we have:

PROPOSITION 2. A given cubic form $F \in S^3H^\vee$ on a finitely generated free abelian group H is realizable as cup-form of a 1-connected, closed, oriented, 6-dimensional manifold with torsion-free homology if and only if it possesses a characteristic element.

Proof. If (r, H, w, τ, F, p) is an admissible system of invariants, and $W_\circ \in H$ any integral lift of w , then we have $p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2 \pmod{24} \quad \forall x \in H$, i.e. the function $l_{W_\circ}: H \rightarrow \mathbf{Z}$ is linear modulo 24, and W_\circ is therefore characteristic for F . Conversely, suppose $W_\circ \in H$ is a characteristic element for a cubic form $F \in S^3H^\vee$; let $w := \bar{W}_\circ(\text{mod } 2)$, $r := 0$.

By the main lemma we have to construct linear forms $p, T \in H^\vee$, such that

- i) $W_\circ^3 \equiv (p + 24T)(W_\circ) \pmod{48}$
- ii) $p(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24} \quad \forall x \in H$.

The function $l_{W_\circ} : H \rightarrow \mathbf{Z}, l_{W_\circ}(x) = 4x^3 + 6x^2 W_\circ + 3x W_\circ^2$ is linear modulo 24 since W_\circ is a characteristic element for F : we therefore choose a linear form $p_\circ \in H^\vee$ with $p_\circ(x) \equiv l_{W_\circ}(x) \pmod{24} \quad \forall x \in H$. Substituting $x = W_\circ$ we find $p_\circ(W_\circ) \equiv 13 W_\circ^3 \pmod{24}$; but since W_\circ is characteristic we have $W_\circ^3 \equiv 0 \pmod{2}$, thus $p_\circ(W_\circ) \equiv W_\circ^3 \pmod{24}$. Write $p_\circ(W_\circ) = W_\circ^3 + 24k$ for some $k \in \mathbf{Z}$.

case 1) $k \equiv 0 \pmod{2}$: define $p := p_\circ, T := 0$.

case 2) $k \equiv 1 \pmod{2}$: we must find a linear form $T_\circ \in H^\vee$ with $T_\circ(W_\circ) \equiv 1 \pmod{2}$; clearly this can be done if and only if W_\circ is not divisible by 2. If W_\circ were divisible by 2, $W_\circ = 2V_\circ$ for some $V_\circ \in H$, then $2p_\circ(V_\circ) = p_\circ(W_\circ) = W_\circ^3 + 24k = 8V_\circ^3 + 24k$ would give $p_\circ(V_\circ) = 4V_\circ^3 + 12k$; then, using $p_\circ(V_\circ) \equiv 4V_\circ^3 + 6V_\circ^2 W_\circ + 3V_\circ W_\circ^2 \equiv 4V_\circ^3 \pmod{24}$ we would find $k \equiv 0 \pmod{2}$, which is not the case by assumption.

This shows that $F \in S^3 H^\vee$ is realizable by a topological manifold with Pontrjagin class p_\circ and non-vanishing triangulation obstruction $\tau_\circ := \bar{T}_\circ \pmod{2}$. In order to realize F by a smooth manifold, one can take $p := p_\circ + 24T_\circ$, and $\tau := 0$.

REMARK 3. The topological counterpart of the existence of a characteristic element for a given cubic form $F \in S^3 H^\vee$ is the existence of a mod-2 Steenrod-algebra structure, which is a necessary condition for a ring to be a cohomology ring.

The existence and the classification of characteristic elements for a given cubic form is essentially a linear algebra problem over $\mathbf{Z}/_2$. To see this, let $F \in S^3 H^\vee$ be a fixed cubic form on a finitely generated free abelian group H . Associated with F we have a linear map $F^t : H \rightarrow S^2 H^\vee$ sending an element $h \in H$ to the bilinear form $F^t(h) : H \otimes H \rightarrow \mathbf{Z}, (x, y) \rightarrow x \cdot y \cdot h$. Let $\bar{H} := H/_2 H$. $\bar{F} \in S^3 \bar{H}^\vee$ be the reductions of H and F modulo 2, and let $- : H \rightarrow \bar{H}$ be the natural epimorphism. The symmetric trilinear form \bar{F} on the $\mathbf{Z}/_2$ -module \bar{H} defines a natural symmetric bilinear form $q_{\bar{F}} \in S^2 \bar{H}^\vee$ given by $q_{\bar{F}}(\bar{x}, \bar{y}) := \bar{x} \cdot \bar{y} \cdot (\bar{x} + \bar{y})$.

LEMMA 3. $F \in S^3 H^\vee$ admits characteristic elements if and only if $q_{\bar{F}}$ lies in the image of $\bar{F}^t \in \text{Hom}_{\mathbf{Z}}(H, S^2 \bar{H}^\vee)$. The set of all characteristic elements for F is a coset of the form $W_\circ + \text{Ker}(\bar{F}^t)$.

Proof. W_o is characteristic for F if and only if $q_{\bar{F}} = \bar{F}^t(W_o)$.

In terms of a \mathbf{Z} -basis $\{e_1, \dots, e_b\}$ for H the condition $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$ translates into a simple rank condition over $\mathbf{Z}_{/2}$: the $\mathbf{Z}_{/2}$ -rank of the $b \times \binom{b+1}{2}$ -matrix A representing \bar{F}^t must be equal to the $\mathbf{Z}_{/2}$ -rank of the matrix A extended by the column $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i < j \leq b}$

EXAMPLE 3. Let $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be free of rank 2, $F \in S^3 H^\vee$ given by $e_1^3 = a, e_1^2 e_2 = b, e_1 e_2^2 = c, e_2^3 = d$ with $a, b, c, d \in \mathbf{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr’s classification theorem we know that in algebraic terms this means the following: fix a non-negative integer r_o , a finitely generated free abelian group H_o , and a symmetric trilinear form $F_o \in S^3 H_o^\vee$ which admits characteristic elements.

Let $\mathcal{M}(r_o, H_o, F_o)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds X with $b_3(X) = 2r_o$, such that there exists an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$. Denote by $\text{Aut}(F_o)$ the subgroup of \mathbf{Z} -automorphisms of H_o which leave $F_o \in S^3 H_o^\vee$ invariant; $\text{Aut}(F_o)$ acts on pairs $(w, [l]) \in \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]) .$$

Let $\text{Aut}(F_o) \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ be the set of $\text{Aut}(F_o)$ -orbits.

A manifold X in $\mathcal{M}(r_o, H_o, F_o)$ and an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$ yields a well-defined $\text{Aut}(F_o)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \text{ (modulo } \text{Aut}(F_o) \text{) ,}$$

where $T \in H^4(X, \mathbf{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$.

The set of oriented homotopy types $\mathcal{M}(r_o, H_o, F_o) / \simeq$ of manifolds in $\mathcal{M}(r_o, H_o, F_o)$ can now be described in the following way:

PROPOSITION 3. *The assignment $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X) + 24T])$ (modulo $\text{Aut}(F_o)$) defines an injection.*

$$I: \mathcal{M}(r_o, H_o, F_o) / \cong \rightarrow_{\text{Aut}(F_o)} \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}.$$

Proof. Suppose X and X' are manifolds in $\mathcal{M}(r_o, H_o, F_o)$, $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ and $\alpha': H_o \rightarrow H^2(X', \mathbf{Z})$ isomorphisms with $\alpha^*F_X = F_o$ and $(\alpha')^*F_{X'} = F_o$. X and X' have the same image under I iff there exists an automorphism $\gamma \in \text{Aut}(F_o)$ with $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1}w_2(X')$ and $(\gamma^{-1})^*\alpha^*[p_1(X) + 24T] = (\alpha')^*[p_1(X') + 24T']$. Consider $\beta := \alpha \circ \gamma \circ \alpha^{-1}: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$; β is obviously an isomorphism with $\beta^*F_{X'} = F_X$, $\beta w_2(X) = w_2(X')$, and $\beta^*[p_1(X') + 24T'] = [p_1(X) + 24T]$; but this means that the systems of invariants associated with X and X' are weakly equivalent, and therefore X and X' oriented homotopy equivalent.

A complete description of the set $\mathcal{M}(r_o, H_o, F_o) / \cong$ i.e. of the image of I is only possible if the automorphism group $\text{Aut}(F_o)$ is known; this can be a serious problem, but we will see that the ‘general’ automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in $\mathcal{M}(r_o, H_o, F_o) / \cong$.

PROPOSITION 4. *Fix $r_o \in \mathbf{N}$, a finitely generated free abelian group H_o , and a symmetric trilinear form $F_o \in S^3H_o^\vee$ which admits characteristic elements. Set $b := rk_{\mathbf{Z}}H_o$, $s := rk_{\mathbf{Z}/2}(\bar{F}_o^t)$, and let $t := rk_{\mathbf{Z}/2}(\cdot_{\bar{F}_o})$ be the $\mathbf{Z}/2$ -rank of the $\mathbf{Z}/2$ -linear square map $\cdot_{\bar{F}_o}: \bar{H}_o \rightarrow \bar{H}_o^\vee$ sending $\bar{u} \in \bar{H}_o$ to $\bar{u}^2 \in \bar{H}_o^\vee$. Then $\mathcal{M}(r_o, H_o, F_o) / \cong$ contains at most 2^{2b-s-t} elements.*

Proof. Fix any admissible system of invariants $(r_o, H_o, w_o, \tau_o, F_o, p_o)$ for a manifold in $\mathcal{M}(r_o, H_o, F_o)$. Given (r_o, H_o, F_o) , we know from the last lemma that the possible elements w_o form a coset of $\text{Ker}(\bar{F}_o^t)$ in \bar{H}_o , so that there exist precisely 2^{b-s} such elements. It remains to count the classes $[l] \in H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$, such that the $\text{Aut}(F_o)$ -orbit of $(w_o, [p_o + 24T_o + l])$ lies in the image of I .

To understand the latter condition we fix integral liftings $W_o, \in H_o, T_o \in H_o^\vee$ of w_o and τ_o satisfying the admissibility conditions

- i) $W_o^3 \equiv (p_o + 24T_o)(W_o) \pmod{48}$
- ii) $p_o(x) \equiv 4x^3 + 6x^2W_o + 3xW_o^2 \pmod{24} \quad \forall x \in H_o.$

Clearly the $\text{Aut}(F_o)$ -orbit of $(w_o, [p_o + 24T_o + l])$ lies in the image of I if and only if

i') $W_o^3 \equiv (p_o + 24T_o + l)(W_o) \pmod{48},$

ii') $(p_o + l)(x) \equiv 4x^3 + 6x^2W_o + 3xW_o^2 \pmod{24} \quad \forall x \in H_o,$

which is equivalent to $l(W_o) \equiv 0 \pmod{48}$, and $l \equiv 0 \pmod{24H_o^\vee}$ because of i) and ii).

Now, by definition of the subgroup $U_{F_o} \subset H_o^\vee /_{48H_o^\vee}$ we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & \text{Ker}(\cdot \bar{F}_o) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow \text{Ker}(24 \cdot \bar{F}_o) & \hookrightarrow & H_o /_{2H_o} & \xrightarrow{24 \cdot \bar{F}_o} & U_{F_o} & \rightarrow & 0 \\
 & & \cdot \bar{F}_o \downarrow & & \downarrow & & \\
 0 & \rightarrow & H_o^\vee /_{2H_o^\vee} & \xrightarrow{24} & H_o^\vee /_{48H_o^\vee} & \rightarrow & H_o^\vee /_{24H_o^\vee} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & \text{Coker}(\cdot \bar{F}_o) & \rightarrow & H_o^\vee /_{48H_o^\vee} /_{U_{F_o}} & \rightarrow & H_o^\vee /_{24H_o^\vee} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The number of elements $[l] \in H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ to be counted coincides therefore with the cardinality of the kernel of the map $ev(w_o): \text{Coker}(\cdot \bar{F}_o) \rightarrow \mathbf{Z}_{/2}$ induced by evaluation in w_o . This number is at most $2^{b-t}(2^{b-t-1}$ if $w_o \neq 0$ and $t \neq b$).

COROLLARY 2. *If the $\mathbf{Z}_{/2}$ -rank $s = rk_{\mathbf{Z}_{/2}}(\cdot \bar{F}_o)$ is maximal, then $\mathcal{M}(r_o, H_o, F_o) /_{=}$ contains at most one class.*

Proof. Suppose $\cdot \bar{F}_o: \bar{H}_o \rightarrow \bar{H}_o^\vee$ is surjective; then $\bar{F}_o^t: \bar{H}_o \rightarrow S^2 \bar{H}_o^\vee$ must have a trivial kernel, since $\bar{h}\bar{x}^2 = 0$ for all $\bar{x} \in \bar{H}_o$ implies $\bar{h} = 0$ if every linear form is a square. But this means $s = t = b$, so that $\mathcal{M}(r_o, H_o, F_o) /_{=}$ has at most one element.

EXAMPLE 4. Let $H_o = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, $e_1^3 = a$, $e_1^2e_2 = b$, $e_1e_2^2 = c$, $e_2^3 = d$. If $\bar{b} \equiv \bar{c} \pmod{2}$, and $\bar{a}\bar{d} - \bar{b}\bar{c} \equiv 1 \pmod{2}$, then $\mathcal{M}(r_o, H_o, F_o) /_{=}$ contains precisely one class for every $r_o \geq 0$.