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Artikel: RATIONALITY PROBLEMS FOR K-THEORY AND CHERN-SIMONS INVARIANTS OF HYPERBOLIC 3-MANIFOLDS

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$\mathcal{B}(E)_{\mathbf{Q}}^{H'} = \mathcal{B}(F')_{\mathbf{Q}}$. We thus obtain inclusions $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$, and the reverse inclusions are trivial. □

Proof of Corollary 3.2. The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$ and if $F' \neq F$ then F has strictly more complex embeddings than F' so $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$. Thus, to have $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$ we must have $F = F'$. The claim then follows directly from Theorem B. □

REMARK. We have pointed out at the beginning of sect. 2 that $\mathcal{B}(F)$ could have been replaced by $K_3(F)$ in all our discussions. The analog of Borel's theorem holds for $K_i(F)$ for all $i \equiv 3 \pmod 4$, so the results described above are also valid for these K-groups. When $1 < i \equiv 1 \pmod 4$ Borel's theorem gives a map $K_i(F) \to \mathbf{R}^{r_1 + r_2}$ whose kernel is torsion and whose image is a lattice. The only change is that one obtains $r_1 + \frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$ as the dimensions of the $+$ and $-$ eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if $E \subset \mathbf{C}$ is Galois over $\mathbf{Q}$ with group G and δ is its conjugation then $K_i(E) \otimes \mathbf{R}$ is G-equivariantly isomorphic to $\{\sum r_\gamma \gamma \in \mathbf{R}G \mid r_\gamma = (-1)^{(i-1)/2} r_{\delta\gamma}\}$ for $i > 1$ and odd.

4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that $D_2(z)$ represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer $N \geqslant 3$, the real numbers $D_2(e^{2\pi\sqrt{-1}j/N})$, with j relatively prime to N and $0 < j < N/2$, are linearly independent over the rationals.*

A field homomorphism $\tau: F \to K$ clearly induces a homomorphism on the Bloch groups $\mathcal{B}(F) \to \mathcal{B}(K)$ which, by abuse of notation, will again be denoted by τ.

Given a cyclotomic field $F = \mathbf{Q}(e^{2\pi\sqrt{-1}/N})$, the elements $[e^{2\pi ij/N}]$, with j relatively prime to N and $0 < j < N/2$, form a basis of the Bloch group $\mathcal{B}(F) \otimes \mathbf{Q}$ (see Bloch [2]). Hence Milnor's conjecture can be reformulated that $D_2: \mathcal{B}(F) \to \mathbf{R}$ given on generators by $[z] \mapsto D_2(z)$ is injective for a cyclotomic field F.

Note that for a general number field F the above map D_2 vanishes on $\mathcal{B}_+(F)$. By Corollary 3.2, the following is thus the strongest generalization of Milnor's conjecture that one might hope for.

CONJECTURE 4.1. *If* $F \subset \mathbf{C}$ *is a number field with* $F \cap \mathbf{R}$ *totally real then the map* $D_2 \colon \mathcal{B}(F) \to \mathbf{R}$ *given by* $[z] \mapsto D_2(z)$ *is injective.*

In the special case that F is Galois over $\mathbf{Q}$ the condition $F \cap \mathbf{R}$ totally real says that F is a CM-field. In this case we have the following proposition (cf. Prop. 7.2.5 of [16])

PROPOSITION 4.2. *Suppose that* F *is a Galois CM-field over* $\mathbf{Q}$. *If for one complex embedding* $\tau : F \hookrightarrow \mathbf{C}$, *the map* $D_2 \circ \tau$ *is injective (that is, Conjecture 4.1 holds), then it is for all complex embeddings.*

Proof. Let $\rho : F \hookrightarrow \mathbf{C}$ be another embedding of F. There exists $\gamma \in \mathrm{Gal}(F/\mathbf{Q})$ such that $\rho = \tau\gamma$. Let $\omega \in B(F) \otimes \mathbf{Q}$. If

$$D_2 \circ \rho(\omega) = D_2 \circ \tau(\gamma(\omega)) = 0\,,$$

then by the injectivity of $D_2 \circ \tau$, $\gamma(\omega) = 0$. Since $\gamma \colon \mathcal{B}(F) \to \mathcal{B}(F)$ is clearly an isomorphism, it follows that $\omega = 0$ in $\mathcal{B}(F) \otimes \mathbf{Q}$. Hence, $D_2 \circ \rho$ is also injective. □

The map $D_2 \colon \mathcal{B}(\mathbf{C}) \to \mathbf{R}$ is the imaginary part of the more general *Bloch* map

$$\rho \colon \mathcal{B}(\mathbf{C}) \to \mathbf{C}/\mathbf{Q}(2)\,,$$

where the notation $\mathbf{Q}(k)$ denotes the subgroup $(2\pi\sqrt{-1})^k\mathbf{Q}$ of $\mathbf{C}$. The definition of ρ is given as follows:

For $z \in \mathbf{C} - \{0, 1\}$, define

$$\rho(z) = \log z \wedge \log(1-z) + 2\pi\sqrt{-1}$$
$$\wedge \frac{1}{2\pi\sqrt{-1}}(\ln_2(1-z) - \ln_2(z) - \pi^2/6) \in \wedge^2_{\mathbf{Z}}\mathbf{C}\,.$$

See section 4 of [6] or [8] for the meaning of this map and for further details. The exact formulae given here is borrowed from [8]. This map obviously induces a map

$$\rho \colon \mathcal{A}(\mathbf{C}) \to \wedge^2_{\mathbf{Z}}\mathbf{C}\,.$$

It turns out that ρ vanishes on the 5-term relation, hence it induces a map

$$\rho \colon \mathcal{B}(\mathbf{C}) \to \wedge^2_{\mathbf{Z}}\mathbf{C}\,.$$

Finally, it follows from the fact that every element α of $\mathscr{B}(\mathbf{C})$ satisfies the relation $\mu(\alpha) = 0$ that the image of this last map lies in the kernel of the map

$$e : \wedge_{\mathbf{Z}}^2 \mathbf{C} \overset{\wedge^2 \exp}{\to} \wedge_{\mathbf{Z}}^2 \mathbf{C}^* .$$

The kernel of e is $\mathbf{C}/\mathbf{Q}(2)$. Hence this induces the Bloch map.

Ramakrishnan [16] generalized Milnor's conjecture in the following form[1]).

RAMAKRISHNAN'S CONJECTURE. *For every subfield* $F \overset{\sigma}{\hookrightarrow} \mathbf{C}$, *the map*

$$\mathscr{B}(F) \otimes \mathbf{Q} \overset{\sigma}{\to} \mathscr{B}(\mathbf{C}) \otimes \mathbf{Q} \to \mathbf{C}/\mathbf{Q}(2)$$

is injective.

The Bloch-Wigner function D_2 is the imaginary part of the Bloch map ρ, and it vanishes identically on $\mathscr{B}_+(k)$. On the other hand, it follows from a routine calculation that the real part of the Bloch map vanishes identically on $\mathscr{B}_-(k)$.

In particular, ρ just reduces to D_2 if $\mathscr{B}_-(F) = \mathscr{B}(F)_{\mathbf{Q}}$. By Corollary 3.2 we thus have

PROPOSITION 4.3. *If* $F \subset \mathbf{C}$ *is a CM-embedded field, then the Ramakrishnan Conjecture for this particular embedding of* F *is equivalent to Conjecture 4.1.* □

On the other hand

PROPOSITION 4.4. *The truth of the Ramakrishnan Conjecture for a field* $E = \bar{E} \subset \mathbf{C}$ *would imply Conjecture 4.1 for any subfield of* E.

Proof. Since the real and imaginary parts of ρ vanish on $\mathscr{B}_-(E)$ and $\mathscr{B}_+(E)$ respectively, the Ramakrishnan conjecture for E is equivalent to the conjecture that the kernel of the real part of ρ is exactly $\mathscr{B}_-(E)$ and the kernel of the imaginary part, that is $Ker(D_2)$, is exactly $\mathscr{B}_+(E)$. Thus D_2 would have zero kernel on $\mathscr{B}(F)_{\mathbf{Q}}$ for any subfield F of E satisfying $\mathscr{B}(F)_{\mathbf{Q}} \cap \mathscr{B}_+(E) = \{0\}$, which is equivalent to the condition of Conjecture 4.1 by Corollary 3.2. □

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[1]) Both Ramakrishnan's conjecture and Milnor's original conjecture are more general in that they apply to all the odd-degree higher K-groups. We refer to [16] for more details. Likewise, Conjecture 4.1 can be stated for higher K-groups in similar fashion.