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$\mathcal{B}(E)_{\mathbf{Q}}^{H'} = \mathcal{B}(F')_{\mathbf{Q}}$ . We thus obtain inclusions  $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$ , and the reverse inclusions are trivial.  $\square$

*Proof of Corollary 3.2.* The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that  $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$  and if  $F' \neq F$  then  $F$  has strictly more complex embeddings than  $F'$  so  $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$ . Thus, to have  $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$  we must have  $F = F'$ . The claim then follows directly from Theorem B.  $\square$

REMARK. We have pointed out at the beginning of sect. 2 that  $\mathcal{B}(F)$  could have been replaced by  $K_3(F)$  in all our discussions. The analog of Borel's theorem holds for  $K_i(F)$  for all  $i \equiv 3 \pmod{4}$ , so the results described above are also valid for these  $K$ -groups. When  $1 < i \equiv 1 \pmod{4}$  Borel's theorem gives a map  $K_i(F) \rightarrow \mathbf{R}^{r_1+r_2}$  whose kernel is torsion and whose image is a lattice. The only change is that one obtains  $r_1 + \frac{1}{2}(r_2 + r'_2)$  and  $\frac{1}{2}(r_2 - r'_2)$  as the dimensions of the  $+$  and  $-$  eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if  $E \subset \mathbf{C}$  is Galois over  $\mathbf{Q}$  with group  $G$  and  $\delta$  is its conjugation then  $K_i(E) \otimes \mathbf{R}$  is  $G$ -equivariantly isomorphic to  $\{ \sum r_{\gamma} \gamma \in \mathbf{R}G \mid r_{\gamma} = (-1)^{(i-1)/2} r_{\delta\gamma} \}$  for  $i > 1$  and odd.

#### 4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that  $D_2(z)$  represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer  $N \geq 3$ , the real numbers  $D_2(e^{2\pi i \frac{-1+j}{N}})$ , with  $j$  relatively prime to  $N$  and  $0 < j < N/2$ , are linearly independent over the rationals.*

A field homomorphism  $\tau: F \rightarrow K$  clearly induces a homomorphism on the Bloch groups  $\mathcal{B}(F) \rightarrow \mathcal{B}(K)$  which, by abuse of notation, will again be denoted by  $\tau$ .

Given a cyclotomic field  $F = \mathbf{Q}(e^{2\pi i \frac{-1}{N}})$ , the elements  $[e^{2\pi i j/N}]$ , with  $j$  relatively prime to  $N$  and  $0 < j < N/2$ , form a basis of the Bloch group  $\mathcal{B}(F) \otimes \mathbf{Q}$  (see Bloch [2]). Hence Milnor's conjecture can be reformulated that  $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$  given on generators by  $[z] \mapsto D_2(z)$  is injective for a cyclotomic field  $F$ .

Note that for a general number field  $F$  the above map  $D_2$  vanishes on  $\mathcal{B}_+(F)$ . By Corollary 3.2, the following is thus the strongest generalization of Milnor's conjecture that one might hope for.

CONJECTURE 4.1. *If  $F \subset \mathbf{C}$  is a number field with  $F \cap \mathbf{R}$  totally real then the map  $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$  given by  $[z] \mapsto D_2(z)$  is injective.*

In the special case that  $F$  is Galois over  $\mathbf{Q}$  the condition  $F \cap \mathbf{R}$  totally real says that  $F$  is a CM-field. In this case we have the following proposition (cf. Prop. 7.2.5 of [16])

PROPOSITION 4.2. *Suppose that  $F$  is a Galois CM-field over  $\mathbf{Q}$ . If for one complex embedding  $\tau: F \hookrightarrow \mathbf{C}$ , the map  $D_2 \circ \tau$  is injective (that is, Conjecture 4.1 holds), then it is for all complex embeddings.*

*Proof.* Let  $\rho: F \hookrightarrow \mathbf{C}$  be another embedding of  $F$ . There exists  $\gamma \in \text{Gal}(F/\mathbf{Q})$  such that  $\rho = \tau\gamma$ . Let  $\omega \in B(F) \otimes \mathbf{Q}$ . If

$$D_2 \circ \rho(\omega) = D_2 \circ \tau(\gamma(\omega)) = 0,$$

then by the injectivity of  $D_2 \circ \tau$ ,  $\gamma(\omega) = 0$ . Since  $\gamma: \mathcal{B}(F) \rightarrow \mathcal{B}(F)$  is clearly an isomorphism, it follows that  $\omega = 0$  in  $\mathcal{B}(F) \otimes \mathbf{Q}$ . Hence,  $D_2 \circ \rho$  is also injective.  $\square$

The map  $D_2: \mathcal{B}(\mathbf{C}) \rightarrow \mathbf{R}$  is the imaginary part of the more general Bloch map

$$\rho: \mathcal{B}(\mathbf{C}) \rightarrow \mathbf{C}/\mathbf{Q}(2),$$

where the notation  $\mathbf{Q}(k)$  denotes the subgroup  $(2\pi\sqrt{-1})^k \mathbf{Q}$  of  $\mathbf{C}$ . The definition of  $\rho$  is given as follows:

For  $z \in \mathbf{C} - \{0, 1\}$ , define

$$\begin{aligned} \rho(z) = & \log z \wedge \log(1-z) + 2\pi\sqrt{-1} \\ & \wedge \frac{1}{2\pi\sqrt{-1}} (\ln_2(1-z) - \ln_2(z) - \pi^2/6) \in \wedge^2_{\mathbf{Z}} \mathbf{C}. \end{aligned}$$

See section 4 of [6] or [8] for the meaning of this map and for further details. The exact formulae given here is borrowed from [8]. This map obviously induces a map

$$\rho: \mathcal{A}(\mathbf{C}) \rightarrow \wedge^2_{\mathbf{Z}} \mathbf{C}.$$

It turns out that  $\rho$  vanishes on the 5-term relation, hence it induces a map

$$\rho: \mathcal{B}(\mathbf{C}) \rightarrow \wedge^2_{\mathbf{Z}} \mathbf{C}.$$

Finally, it follows from the fact that every element  $\alpha$  of  $\mathcal{B}(\mathbf{C})$  satisfies the relation  $\mu(\alpha) = 0$  that the image of this last map lies in the kernel of the map

$$e: \wedge_{\mathbf{Z}}^2 \mathbf{C} \xrightarrow{\wedge^2 \exp} \wedge_{\mathbf{Z}}^2 \mathbf{C}^*.$$

The kernel of  $e$  is  $\mathbf{C}/\mathbf{Q}(2)$ . Hence this induces the Bloch map.

Ramakrishnan [16] generalized Milnor's conjecture in the following form<sup>1)</sup>.

RAMAKRISHNAN'S CONJECTURE. *For every subfield  $F \xrightarrow{\sigma} \mathbf{C}$ , the map*

$$\mathcal{B}(F) \otimes \mathbf{Q} \xrightarrow{\sigma} \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q} \rightarrow \mathbf{C}/\mathbf{Q}(2)$$

*is injective.*

The Bloch-Wigner function  $D_2$  is the imaginary part of the Bloch map  $\rho$ , and it vanishes identically on  $\mathcal{B}_+(k)$ . On the other hand, it follows from a routine calculation that the real part of the Bloch map vanishes identically on  $\mathcal{B}_-(k)$ .

In particular,  $\rho$  just reduces to  $D_2$  if  $\mathcal{B}_-(F) = \mathcal{B}(F)_{\mathbf{Q}}$ . By Corollary 3.2 we thus have

PROPOSITION 4.3. *If  $F \subset \mathbf{C}$  is a CM-embedded field, then the Ramakrishnan Conjecture for this particular embedding of  $F$  is equivalent to Conjecture 4.1.  $\square$*

On the other hand

PROPOSITION 4.4. *The truth of the Ramakrishnan Conjecture for a field  $E = \bar{E} \subset \mathbf{C}$  would imply Conjecture 4.1 for any subfield of  $E$ .*

*Proof.* Since the real and imaginary parts of  $\rho$  vanish on  $\mathcal{B}_-(E)$  and  $\mathcal{B}_+(E)$  respectively, the Ramakrishnan conjecture for  $E$  is equivalent to the conjecture that the kernel of the real part of  $\rho$  is exactly  $\mathcal{B}_-(E)$  and the kernel of the imaginary part, that is  $\text{Ker}(D_2)$ , is exactly  $\mathcal{B}_+(E)$ . Thus  $D_2$  would have zero kernel on  $\mathcal{B}(F)_{\mathbf{Q}}$  for any subfield  $F$  of  $E$  satisfying  $\mathcal{B}(F)_{\mathbf{Q}} \cap \mathcal{B}_+(E) = \{0\}$ , which is equivalent to the condition of Conjecture 4.1 by Corollary 3.2.  $\square$

<sup>1)</sup> Both Ramakrishnan's conjecture and Milnor's original conjecture are more general in that they apply to all the odd-degree higher  $K$ -groups. We refer to [16] for more details. Likewise, Conjecture 4.1 can be stated for higher  $K$ -groups in similar fashion.