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$\mathcal{B}(E)^{H'}_{\mathbf{Q}} = \mathcal{B}(F')_{\mathbf{Q}}$. We thus obtain inclusions $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$, and the reverse inclusions are trivial. \square

Proof of Corollary 3.2. The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$ and if $F' \neq F$ then F has strictly more complex embeddings than F' so $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$. Thus, to have $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$ we must have $F = F'$. The claim then follows directly from Theorem B. \square

REMARK. We have pointed out at the beginning of sect. 2 that $\mathcal{B}(F)$ could have been replaced by $K_3(F)$ in all our discussions. The analog of Borel's theorem holds for $K_i(F)$ for all $i \equiv 3 \pmod{4}$, so the results described above are also valid for these K -groups. When $1 < i \equiv 1 \pmod{4}$ Borel's theorem gives a map $K_i(F) \rightarrow \mathbf{R}^{r_1+r_2}$ whose kernel is torsion and whose image is a lattice. The only change is that one obtains $r_1 + \frac{1}{2}(r_2 + r'_2)$ and $\frac{1}{2}(r_2 - r'_2)$ as the dimensions of the + and - eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if $E \subset \mathbf{C}$ is Galois over \mathbf{Q} with group G and δ is its conjugation then $K_i(E) \otimes \mathbf{R}$ is G -equivariantly isomorphic to $\{ \sum r_{\gamma} \gamma \in \mathbf{R}G \mid r_{\gamma} = (-1)^{(i-1)/2} r_{\delta\gamma} \}$ for $i > 1$ and odd.

4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that $D_2(z)$ represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer $N \geq 3$, the real numbers $D_2(e^{2\pi i \frac{-1}{N} j})$, with j relatively prime to N and $0 < j < N/2$, are linearly independent over the rationals.*

A field homomorphism $\tau: F \rightarrow K$ clearly induces a homomorphism on the Bloch groups $\mathcal{B}(F) \rightarrow \mathcal{B}(K)$ which, by abuse of notation, will again be denoted by τ .

Given a cyclotomic field $F = \mathbf{Q}(e^{2\pi i \frac{-1}{N}})$, the elements $[e^{2\pi i j/N}]$, with j relatively prime to N and $0 < j < N/2$, form a basis of the Bloch group $\mathcal{B}(F) \otimes \mathbf{Q}$ (see Bloch [2]). Hence Milnor's conjecture can be reformulated that $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$ given on generators by $[z] \mapsto D_2(z)$ is injective for a cyclotomic field F .

Note that for a general number field F the above map D_2 vanishes on $\mathcal{B}_+(F)$. By Corollary 3.2, the following is thus the strongest generalization of Milnor's conjecture that one might hope for.

CONJECTURE 4.1. *If $F \subset \mathbf{C}$ is a number field with $F \cap \mathbf{R}$ totally real then the map $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$ given by $[z] \mapsto D_2(z)$ is injective.*

In the special case that F is Galois over \mathbf{Q} the condition $F \cap \mathbf{R}$ totally real says that F is a CM-field. In this case we have the following proposition (cf. Prop. 7.2.5 of [16])

PROPOSITION 4.2. *Suppose that F is a Galois CM-field over \mathbf{Q} . If for one complex embedding $\tau: F \hookrightarrow \mathbf{C}$, the map $D_2 \circ \tau$ is injective (that is, Conjecture 4.1 holds), then it is for all complex embeddings.*

Proof. Let $\rho: F \hookrightarrow \mathbf{C}$ be another embedding of F . There exists $\gamma \in \text{Gal}(F/\mathbf{Q})$ such that $\rho = \tau\gamma$. Let $\omega \in B(F) \otimes \mathbf{Q}$. If

$$D_2 \circ \rho(\omega) = D_2 \circ \tau(\gamma(\omega)) = 0,$$

then by the injectivity of $D_2 \circ \tau$, $\gamma(\omega) = 0$. Since $\gamma: \mathcal{B}(F) \rightarrow \mathcal{B}(F)$ is clearly an isomorphism, it follows that $\omega = 0$ in $\mathcal{B}(F) \otimes \mathbf{Q}$. Hence, $D_2 \circ \rho$ is also injective. \square

The map $D_2: \mathcal{B}(\mathbf{C}) \rightarrow \mathbf{R}$ is the imaginary part of the more general *Bloch* map

$$\rho: \mathcal{B}(\mathbf{C}) \rightarrow \mathbf{C}/\mathbf{Q}(2),$$

where the notation $\mathbf{Q}(k)$ denotes the subgroup $(2\pi\sqrt{-1})^k \mathbf{Q}$ of \mathbf{C} . The definition of ρ is given as follows:

For $z \in \mathbf{C} - \{0, 1\}$, define

$$\begin{aligned} \rho(z) &= \log z \wedge \log(1-z) + 2\pi\sqrt{-1} \\ &\wedge \frac{1}{2\pi\sqrt{-1}} (\ln_2(1-z) - \ln_2(z) - \pi^2/6) \in \wedge_z^2 \mathbf{C}. \end{aligned}$$

See section 4 of [6] or [8] for the meaning of this map and for further details. The exact formulae given here is borrowed from [8]. This map obviously induces a map

$$\rho: \mathcal{A}(\mathbf{C}) \rightarrow \wedge_z^2 \mathbf{C}.$$

It turns out that ρ vanishes on the 5-term relation, hence it induces a map

$$\rho: \mathcal{B}(\mathbf{C}) \rightarrow \wedge_z^2 \mathbf{C}.$$

Finally, it follows from the fact that every element α of $\mathcal{B}(\mathbf{C})$ satisfies the relation $\mu(\alpha) = 0$ that the image of this last map lies in the kernel of the map

$$e: \wedge_Z^2 \mathbf{C} \xrightarrow{\wedge^2 \exp} \wedge_Z^2 \mathbf{C}^*.$$

The kernel of e is $\mathbf{C}/\mathbf{Q}(2)$. Hence this induces the Bloch map.

Ramakrishnan [16] generalized Milnor's conjecture in the following form¹⁾.

RAMAKRISHNAN'S CONJECTURE. *For every subfield $F \xrightarrow{\xi} \mathbf{C}$, the map*

$$\mathcal{B}(F) \otimes \mathbf{Q} \xrightarrow{\circ} \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q} \rightarrow \mathbf{C}/\mathbf{Q}(2)$$

is injective.

The Bloch-Wigner function D_2 is the imaginary part of the Bloch map ρ , and it vanishes identically on $\mathcal{B}_+(k)$. On the other hand, it follows from a routine calculation that the real part of the Bloch map vanishes identically on $\mathcal{B}_-(k)$.

In particular, ρ just reduces to D_2 if $\mathcal{B}_-(F) = \mathcal{B}(F)_{\mathbf{Q}}$. By Corollary 3.2 we thus have

PROPOSITION 4.3. *If $F \subset \mathbf{C}$ is a CM-embedded field, then the Ramakrishnan Conjecture for this particular embedding of F is equivalent to Conjecture 4.1. \square*

On the other hand

PROPOSITION 4.4. *The truth of the Ramakrishnan Conjecture for a field $E = \bar{E} \subset \mathbf{C}$ would imply Conjecture 4.1 for any subfield of E .*

Proof. Since the real and imaginary parts of ρ vanish on $\mathcal{B}_-(E)$ and $\mathcal{B}_+(E)$ respectively, the Ramakrishnan conjecture for E is equivalent to the conjecture that the kernel of the real part of ρ is exactly $\mathcal{B}_-(E)$ and the kernel of the imaginary part, that is $\text{Ker}(D_2)$, is exactly $\mathcal{B}_+(E)$. Thus D_2 would have zero kernel on $\mathcal{B}(F)_{\mathbf{Q}}$ for any subfield F of E satisfying $\mathcal{B}(F)_{\mathbf{Q}} \cap \mathcal{B}_+(E) = \{0\}$, which is equivalent to the condition of Conjecture 4.1 by Corollary 3.2. \square

¹⁾ Both Ramakrishnan's conjecture and Milnor's original conjecture are more general in that they apply to all the odd-degree higher K -groups. We refer to [16] for more details. Likewise, Conjecture 4.1 can be stated for higher K -groups in similar fashion.