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3. PROOF AND GENERALIZATION OF THEOREM B

We shall give a proof of Theorem B based directly on Borel's theorem. A different proof can be given using Proposition 2.1 and its consequence Theorem 3.1.

We denote the complex conjugation in \mathbf{C} by δ . Let F be a fixed non-real subfield of \mathbf{C} that is stable under complex conjugation, i.e., $\delta(F) = F$. Assume F is a finite extension field of \mathbf{Q} . Let r_2 be as in Borel's Theorem. Then we may list all the complex (non-real) embeddings of F into \mathbf{C} as $\tau_1, \delta\tau_1, \dots, \tau_{r_2}, \delta\tau_{r_2}$. Let r'_2 be the number of conjugate pairs that commutes with δ , i.e., $\tau_i\delta = \delta\tau_i$. Renumbering if necessary, we may assume $\tau_1, \dots, \tau_{r'_2}$ are the ones that commute with δ . Note that by our assumption on F , r'_2 is at least one. The rest of the τ 's won't commute with δ , therefore τ_i and $\tau_i\delta, i > r'_2$ will be in different conjugate pairs (we use here that F is non-real). So renumbering if necessary, we may assume

$$\tau_1, \dots, \tau_{r'_2}, \tau_{r'_2+1}, \tau_{r'_2+1}\delta, \dots, \tau_m, \tau_m\delta$$

gives exactly one representative from each conjugate pair of embeddings of F into \mathbf{C} , where $m = r'_2 + (r_2 - r'_2)/2$.

The complex conjugation on F induces an involution on $\mathcal{B}(F)$. Let $\mathcal{B}_+(F)$ and $\mathcal{B}_-(F)$ be the \pm eigenspace of $\mathcal{B}(F)_{\mathbf{Q}} = \mathcal{B}(F) \otimes \mathbf{Q}$. By Borel's Theorem, $\mathcal{B}(F)_{\mathbf{Q}}$ has a \mathbf{Q} -basis $\alpha_1, \dots, \alpha_{r_2}$. Let

$$u_i = \alpha_i - \delta\alpha_i, \quad v_i = \alpha_i + \delta\alpha_i, \quad 1 \leq i \leq r_2.$$

Then u_i 's and v_i 's span $\mathcal{B}_-(F)$ and $\mathcal{B}_+(F)$ respectively. Together, they span $\mathcal{B}(F)_{\mathbf{Q}}$. Hence by Borel's Theorem, we know that the matrix

$$\begin{pmatrix} D_2(\tau_1(u_1)) & \cdots & D_2(\tau_m(u_1)) & D_2(\tau_{r'_2+1}\delta(u_1)) & \cdots & D_2(\tau_m\delta(u_1)) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_2(\tau_{r_2}(u_{r_2})) & \cdots & D_2(\tau_m(u_{r_2})) & D_2(\tau_{r'_2+1}\delta(u_{r_2})) & \cdots & D_2(\tau_m\delta(u_{r_2})) \\ D_2(\tau_1(v_1)) & \cdots & D_2(\tau_m(v_1)) & D_2(\tau_{r'_2+1}\delta(v_1)) & \cdots & D_2(\tau_m\delta(v_1)) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_2(\tau_{r_2}(v_{r_2})) & \cdots & D_2(\tau_m(v_{r_2})) & D_2(\tau_{r'_2+1}\delta(v_{r_2})) & \cdots & D_2(\tau_m\delta(v_{r_2})) \end{pmatrix}$$

has rank r_2 . Note that because the first r'_2 embeddings commute with δ , the entries of the last r_2 rows of the first r_2 columns are all 0's. Also, it follows from the equation $\delta(u_i) = -\delta(u_i)$ and $\delta(v_i) = \delta(v_i)$, this matrix has the following block form

$$\begin{pmatrix} A_{r_2 \times r'_2} & B_{r_2 \times (r_2 - r'_2)/2} & -B_{r_2 \times (r_2 - r'_2)/2} \\ 0 & C_{r_2 \times (r_2 - r'_2)/2} & C_{r_2 \times (r_2 - r'_2)/2} \end{pmatrix}$$

So the matrix A has to have rank r'_2 . For the last $(r_2 - r'_2)$ -columns to have rank $r_2 - r'_2$, the matrices B and C must both have maximal possible rank, that is, $(r_2 - r'_2)/2$. Since by Borel's Theorem,

$$\text{rank } C = \text{rank } \mathcal{B}_+(F),$$

and $\text{rank } \mathcal{B}_+(F) + \text{rank } \mathcal{B}_-(F) = r_2$, Theorem B follows. \square

We can also describe the situation when $F \subset \mathbf{C}$ is a number field that is not stable under conjugation. If $E \subset \mathbf{C}$ is any number field containing F with $E = \bar{E}$ then $\mathcal{B}_+(E)$ and $\mathcal{B}_-(E)$ are defined, so we can form

$$\mathcal{B}_+(F) := \mathcal{B}_+(E) \cap \mathcal{B}(F)_{\mathbf{Q}} \quad \text{and} \quad \mathcal{B}_-(F) := \mathcal{B}_-(E) \cap \mathcal{B}(F)_{\mathbf{Q}}.$$

These subgroups are independent of the choice of E , but in general they will not sum to $\mathcal{B}(F)_{\mathbf{Q}}$.

Denote $F_{\mathbf{R}} = F \cap \mathbf{R}$ and let $F' = F \cap \bar{F}$. Clearly F' contains $F_{\mathbf{R}}$, and $F_{\mathbf{R}}$ must be the fixed field of conjugation on F' . Thus either $F' = F_{\mathbf{R}}$ or F' is an imaginary quadratic extension of $F_{\mathbf{R}}$. Now F' is a field to which Theorem B applies, so $\mathcal{B}(F')_{\mathbf{Q}} = \mathcal{B}_+(F') \oplus \mathcal{B}_-(F')$, with the ranks of the summands given by Theorem B.

THEOREM 3.1. $\mathcal{B}_-(F) = \mathcal{B}_-(F')$ and $\mathcal{B}_+(F) = \mathcal{B}_+(F') = \mathcal{B}(F_{\mathbf{R}})_{\mathbf{Q}}$.

COROLLARY 3.2. $\mathcal{B}_+(F)$ is trivial if and only if $F_{\mathbf{R}}$ is totally real.

$\mathcal{B}_-(F)$ is trivial if and only if $F' = F_{\mathbf{R}}$.

$\mathcal{B}_-(F) = \mathcal{B}(F)_{\mathbf{Q}}$ if and only if $F = F'$ and $F_{\mathbf{R}}$ is totally real; that is either F is totally real or the embedding $F \hookrightarrow \mathbf{C}$ is a CM-embedding.

Proof of Theorem 3.1. We work in a Galois superfield E of F and identify Bloch groups with their images in $\mathcal{B}(E)_{\mathbf{Q}}$. Let $G = \text{Gal}(E/\mathbf{Q})$, so $H = \text{Gal}(E/F) \subset G$ is the subgroup which fixes F . We fix an embedding $E \subset \mathbf{C}$ extending the given embedding of F and denote complex conjugation for this embedding by δ . Then the subgroup $H_{\mathbf{R}}$ generated by H and δ is $\text{Gal}(E/F_{\mathbf{R}})$, so it follows from Proposition 2.1 that $\mathcal{B}_+(F) = \mathcal{B}(E^H)_{\mathbf{Q}} \cap \mathcal{B}(E)_{\mathbf{Q}}^{\delta} = (\mathcal{B}(E)_{\mathbf{Q}}^H)^{\delta} = \mathcal{B}(E)_{\mathbf{Q}}^{H_{\mathbf{R}}} = \mathcal{B}(F_{\mathbf{R}})_{\mathbf{Q}}$. Moreover, $\mathcal{B}_-(F)$ is fixed by both H and $\delta H \delta$ and hence by the group H' that they generate. But H' is the Galois group in E of $F \cap \delta(F) = F \cap \bar{F} = F'$. Thus $\mathcal{B}_-(F)$ is in

$\mathcal{B}(E)^{H'}_{\mathbf{Q}} = \mathcal{B}(F')_{\mathbf{Q}}$. We thus obtain inclusions $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$, and the reverse inclusions are trivial. \square

Proof of Corollary 3.2. The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$ and if $F' \neq F$ then F has strictly more complex embeddings than F' so $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$. Thus, to have $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$ we must have $F = F'$. The claim then follows directly from Theorem B. \square

REMARK. We have pointed out at the beginning of sect. 2 that $\mathcal{B}(F)$ could have been replaced by $K_3(F)$ in all our discussions. The analog of Borel's theorem holds for $K_i(F)$ for all $i \equiv 3 \pmod{4}$, so the results described above are also valid for these K -groups. When $1 < i \equiv 1 \pmod{4}$ Borel's theorem gives a map $K_i(F) \rightarrow \mathbf{R}^{r_1+r_2}$ whose kernel is torsion and whose image is a lattice. The only change is that one obtains $r_1 + \frac{1}{2}(r_2 + r'_2)$ and $\frac{1}{2}(r_2 - r'_2)$ as the dimensions of the + and - eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if $E \subset \mathbf{C}$ is Galois over \mathbf{Q} with group G and δ is its conjugation then $K_i(E) \otimes \mathbf{R}$ is G -equivariantly isomorphic to $\{ \sum r_{\gamma} \gamma \in \mathbf{R}G \mid r_{\gamma} = (-1)^{(i-1)/2} r_{\delta\gamma} \}$ for $i > 1$ and odd.

4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that $D_2(z)$ represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer $N \geq 3$, the real numbers $D_2(e^{2\pi i \frac{-1}{N} j})$, with j relatively prime to N and $0 < j < N/2$, are linearly independent over the rationals.*

A field homomorphism $\tau: F \rightarrow K$ clearly induces a homomorphism on the Bloch groups $\mathcal{B}(F) \rightarrow \mathcal{B}(K)$ which, by abuse of notation, will again be denoted by τ .

Given a cyclotomic field $F = \mathbf{Q}(e^{2\pi i \frac{-1}{N}})$, the elements $[e^{2\pi i j/N}]$, with j relatively prime to N and $0 < j < N/2$, form a basis of the Bloch group $\mathcal{B}(F) \otimes \mathbf{Q}$ (see Bloch [2]). Hence Milnor's conjecture can be reformulated that $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$ given on generators by $[z] \mapsto D_2(z)$ is injective for a cyclotomic field F .