

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: RATIONALITY PROBLEMS FOR K-THEORY AND CHERN-SIMONS INVARIANTS OF HYPERBOLIC 3-MANIFOLDS
Autor: Neumann, Walter D. / Yang, Jun
Kapitel: 3. Proof and generalization of Theorem B
DOI: <https://doi.org/10.5169/seals-61828>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 30.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. PROOF AND GENERALIZATION OF THEOREM B

We shall give a proof of Theorem B based directly on Borel's theorem. A different proof can be given using Proposition 2.1 and its consequence Theorem 3.1.

We denote the complex conjugation in \mathbf{C} by δ . Let F be a fixed non-real subfield of \mathbf{C} that is stable under complex conjugation, i.e., $\delta(F) = F$. Assume F is a finite extension field of \mathbf{Q} . Let r_2 be as in Borel's Theorem. Then we may list all the complex (non-real) embeddings of F into \mathbf{C} as $\tau_1, \delta\tau_1, \dots, \tau_{r_2}, \delta\tau_{r_2}$. Let r'_2 be the number of conjugate pairs that commutes with δ , i.e., $\tau_i\delta = \delta\tau_i$. Renumbering if necessary, we may assume $\tau_1, \dots, \tau_{r'_2}$ are the ones that commute with δ . Note that by our assumption on F , r'_2 is at least one. The rest of the τ 's won't commute with δ , therefore τ_i and $\tau_i\delta, i > r'_2$ will be in different conjugate pairs (we use here that F is non-real). So renumbering if necessary, we may assume

$$\tau_1, \dots, \tau_{r'_2}, \tau_{r'_2+1}, \tau_{r'_2+1}\delta, \dots, \tau_m, \tau_m\delta$$

gives exactly one representative from each conjugate pair of embeddings of F into \mathbf{C} , where $m = r'_2 + (r_2 - r'_2)/2$.

The complex conjugation on F induces an involution on $\mathcal{B}(F)$. Let $\mathcal{B}_+(F)$ and $\mathcal{B}_-(F)$ be the \pm eigenspace of $\mathcal{B}(F)_{\mathbf{Q}} = \mathcal{B}(F) \otimes \mathbf{Q}$. By Borel's Theorem, $\mathcal{B}(F)_{\mathbf{Q}}$ has a \mathbf{Q} -basis $\alpha_1, \dots, \alpha_{r_2}$. Let

$$u_i = \alpha_i - \delta\alpha_i, \quad v_i = \alpha_i + \delta\alpha_i, \quad 1 \leq i \leq r_2.$$

Then u_i 's and v_i 's span $\mathcal{B}_-(F)$ and $\mathcal{B}_+(F)$ respectively. Together, they span $\mathcal{B}(F)_{\mathbf{Q}}$. Hence by Borel's Theorem, we know that the matrix

$$\begin{pmatrix} D_2(\tau_1(u_1)) & \cdots & D_2(\tau_m(u_1)) & D_2(\tau_{r'_2+1}\delta(u_1)) & \cdots & D_2(\tau_m\delta(u_1)) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_2(\tau_{r_2}(u_{r_2})) & \cdots & D_2(\tau_m(u_{r_2})) & D_2(\tau_{r'_2+1}\delta(u_{r_2})) & \cdots & D_2(\tau_m\delta(u_{r_2})) \\ D_2(\tau_1(v_1)) & \cdots & D_2(\tau_m(v_1)) & D_2(\tau_{r'_2+1}\delta(v_1)) & \cdots & D_2(\tau_m\delta(v_1)) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_2(\tau_{r_2}(v_{r_2})) & \cdots & D_2(\tau_m(v_{r_2})) & D_2(\tau_{r'_2+1}\delta(v_{r_2})) & \cdots & D_2(\tau_m\delta(v_{r_2})) \end{pmatrix}$$

has rank r_2 . Note that because the first r'_2 embeddings commute with δ , the entries of the last r_2 rows of the first r_2 columns are all 0's. Also, it follows from the equation $\delta(u_i) = -\delta(u_i)$ and $\delta(v_i) = \delta(v_i)$, this matrix has the following block form

$$\begin{pmatrix} A_{r_2 \times r'_2} & B_{r_2 \times (r_2 - r'_2)/2} & -B_{r_2 \times (r_2 - r'_2)/2} \\ 0 & C_{r_2 \times (r_2 - r'_2)/2} & C_{r_2 \times (r_2 - r'_2)/2} \end{pmatrix}$$

So the matrix A has to have rank r'_2 . For the last $(r_2 - r'_2)$ -columns to have rank $r_2 - r'_2$, the matrices B and C must both have maximal possible rank, that is, $(r_2 - r'_2)/2$. Since by Borel's Theorem,

$$\text{rank } C = \text{rank } \mathcal{B}_+(F),$$

and $\text{rank } \mathcal{B}_+(F) + \text{rank } \mathcal{B}_-(F) = r_2$, Theorem B follows. \square

We can also describe the situation when $F \subset \mathbf{C}$ is a number field that is not stable under conjugation. If $E \subset \mathbf{C}$ is any number field containing F with $E = \bar{E}$ then $\mathcal{B}_+(E)$ and $\mathcal{B}_-(E)$ are defined, so we can form

$$\mathcal{B}_+(F) := \mathcal{B}_+(E) \cap \mathcal{B}(F)_{\mathbf{Q}} \quad \text{and} \quad \mathcal{B}_-(F) := \mathcal{B}_-(E) \cap \mathcal{B}(F)_{\mathbf{Q}}.$$

These subgroups are independent of the choice of E , but in general they will not sum to $\mathcal{B}(F)_{\mathbf{Q}}$.

Denote $F_{\mathbf{R}} = F \cap \mathbf{R}$ and let $F' = F \cap \bar{F}$. Clearly F' contains $F_{\mathbf{R}}$, and $F_{\mathbf{R}}$ must be the fixed field of conjugation on F' . Thus either $F' = F_{\mathbf{R}}$ or F' is an imaginary quadratic extension of $F_{\mathbf{R}}$. Now F' is a field to which Theorem B applies, so $\mathcal{B}(F')_{\mathbf{Q}} = \mathcal{B}_+(F') \oplus \mathcal{B}_-(F')$, with the ranks of the summands given by Theorem B.

THEOREM 3.1. $\mathcal{B}_-(F) = \mathcal{B}_-(F')$ and $\mathcal{B}_+(F) = \mathcal{B}_+(F') = \mathcal{B}(F_{\mathbf{R}})_{\mathbf{Q}}$.

COROLLARY 3.2. $\mathcal{B}_+(F)$ is trivial if and only if $F_{\mathbf{R}}$ is totally real.

$\mathcal{B}_-(F)$ is trivial if and only if $F' = F_{\mathbf{R}}$.

$\mathcal{B}_-(F) = \mathcal{B}(F)_{\mathbf{Q}}$ if and only if $F = F'$ and $F_{\mathbf{R}}$ is totally real; that is either F is totally real or the embedding $F \hookrightarrow \mathbf{C}$ is a CM-embedding.

Proof of Theorem 3.1. We work in a Galois superfield E of F and identify Bloch groups with their images in $\mathcal{B}(E)_{\mathbf{Q}}$. Let $G = \text{Gal}(E/\mathbf{Q})$, so $H = \text{Gal}(E/F) \subset G$ is the subgroup which fixes F . We fix an embedding $E \subset \mathbf{C}$ extending the given embedding of F and denote complex conjugation for this embedding by δ . Then the subgroup $H_{\mathbf{R}}$ generated by H and δ is $\text{Gal}(E/F_{\mathbf{R}})$, so it follows from Proposition 2.1 that $\mathcal{B}_+(F) = \mathcal{B}(E^H)_{\mathbf{Q}} \cap \mathcal{B}(E)_{\mathbf{Q}}^{\delta} = (\mathcal{B}(E)_{\mathbf{Q}}^H)^{\delta} = \mathcal{B}(E)_{\mathbf{Q}}^{H_{\mathbf{R}}} = \mathcal{B}(F_{\mathbf{R}})_{\mathbf{Q}}$. Moreover, $\mathcal{B}_-(F)$ is fixed by both H and $\delta H \delta$ and hence by the group H' that they generate. But H' is the Galois group in E of $F \cap \delta(F) = F \cap \bar{F} = F'$. Thus $\mathcal{B}_-(F)$ is in

$\mathcal{B}(E)_{\mathbf{Q}}^{H'} = \mathcal{B}(F')_{\mathbf{Q}}$. We thus obtain inclusions $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$, and the reverse inclusions are trivial. \square

Proof of Corollary 3.2. The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$ and if $F' \neq F$ then F has strictly more complex embeddings than F' so $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$. Thus, to have $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$ we must have $F = F'$. The claim then follows directly from Theorem B. \square

REMARK. We have pointed out at the beginning of sect. 2 that $\mathcal{B}(F)$ could have been replaced by $K_3(F)$ in all our discussions. The analog of Borel's theorem holds for $K_i(F)$ for all $i \equiv 3 \pmod{4}$, so the results described above are also valid for these K -groups. When $1 < i \equiv 1 \pmod{4}$ Borel's theorem gives a map $K_i(F) \rightarrow \mathbf{R}^{r_1+r_2}$ whose kernel is torsion and whose image is a lattice. The only change is that one obtains $r_1 + \frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$ as the dimensions of the $+$ and $-$ eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if $E \subset \mathbf{C}$ is Galois over \mathbf{Q} with group G and δ is its conjugation then $K_i(E) \otimes \mathbf{R}$ is G -equivariantly isomorphic to $\{ \sum r_{\gamma} \gamma \in \mathbf{R}G \mid r_{\gamma} = (-1)^{(i-1)/2} r_{\delta\gamma} \}$ for $i > 1$ and odd.

4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that $D_2(z)$ represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer $N \geq 3$, the real numbers $D_2(e^{2\pi i \frac{-1-j}{N}})$, with j relatively prime to N and $0 < j < N/2$, are linearly independent over the rationals.*

A field homomorphism $\tau: F \rightarrow K$ clearly induces a homomorphism on the Bloch groups $\mathcal{B}(F) \rightarrow \mathcal{B}(K)$ which, by abuse of notation, will again be denoted by τ .

Given a cyclotomic field $F = \mathbf{Q}(e^{2\pi i \frac{-1}{N}})$, the elements $[e^{2\pi i j/N}]$, with j relatively prime to N and $0 < j < N/2$, form a basis of the Bloch group $\mathcal{B}(F) \otimes \mathbf{Q}$ (see Bloch [2]). Hence Milnor's conjecture can be reformulated that $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$ given on generators by $[z] \mapsto D_2(z)$ is injective for a cyclotomic field F .