

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 41 (1995)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** RATIONALITY PROBLEMS FOR K-THEORY AND CHERN-SIMONS INVARIANTS OF HYPERBOLIC 3-MANIFOLDS  
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**Kapitel:** 3. Proof and generalization of Theorem B  
**DOI:** <https://doi.org/10.5169/seals-61828>

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## 3. PROOF AND GENERALIZATION OF THEOREM B

We shall give a proof of Theorem B based directly on Borel's theorem. A different proof can be given using Proposition 2.1 and its consequence Theorem 3.1.

We denote the complex conjugation in  $\mathbf{C}$  by  $\delta$ . Let  $F$  be a fixed non-real subfield of  $\mathbf{C}$  that is stable under complex conjugation, i.e.,  $\delta(F) = F$ . Assume  $F$  is a finite extension field of  $\mathbf{Q}$ . Let  $r_2$  be as in Borel's Theorem. Then we may list all the complex (non-real) embeddings of  $F$  into  $\mathbf{C}$  as  $\tau_1, \delta\tau_1, \dots, \tau_{r_2}, \delta\tau_{r_2}$ . Let  $r'_2$  be the number of conjugate pairs that commutes with  $\delta$ , i.e.,  $\tau_i\delta = \delta\tau_i$ . Renumbering if necessary, we may assume  $\tau_1, \dots, \tau_{r'_2}$  are the ones that commute with  $\delta$ . Note that by our assumption on  $F$ ,  $r'_2$  is at least one. The rest of the  $\tau$ 's won't commute with  $\delta$ , therefore  $\tau_i$  and  $\tau_i\delta, i > r'_2$  will be in different conjugate pairs (we use here that  $F$  is non-real). So renumbering if necessary, we may assume

$$\tau_1, \dots, \tau_{r'_2}, \tau_{r'_2+1}, \tau_{r'_2+1}\delta, \dots, \tau_m, \tau_m\delta$$

gives exactly one representative from each conjugate pair of embeddings of  $F$  into  $\mathbf{C}$ , where  $m = r'_2 + (r_2 - r'_2)/2$ .

The complex conjugation on  $F$  induces an involution on  $\mathcal{B}(F)$ . Let  $\mathcal{B}_+(F)$  and  $\mathcal{B}_-(F)$  be the  $\pm$  eigenspace of  $\mathcal{B}(F)_{\mathbf{Q}} = \mathcal{B}(F) \otimes \mathbf{Q}$ . By Borel's Theorem,  $\mathcal{B}(F)_{\mathbf{Q}}$  has a  $\mathbf{Q}$ -basis  $\alpha_1, \dots, \alpha_{r_2}$ . Let

$$u_i = \alpha_i - \delta\alpha_i, \quad v_i = \alpha_i + \delta\alpha_i, \quad 1 \leq i \leq r_2.$$

Then  $u_i$ 's and  $v_i$ 's span  $\mathcal{B}_-(F)$  and  $\mathcal{B}_+(F)$  respectively. Together, they span  $\mathcal{B}(F)_{\mathbf{Q}}$ . Hence by Borel's Theorem, we know that the matrix

$$\begin{pmatrix} D_2(\tau_1(u_1)) & \cdots & D_2(\tau_m(u_1)) & D_2(\tau_{r'_2+1}\delta(u_1)) & \cdots & D_2(\tau_m\delta(u_1)) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_2(\tau_{r_2}(u_{r_2})) & \cdots & D_2(\tau_m(u_{r_2})) & D_2(\tau_{r'_2+1}\delta(u_{r_2})) & \cdots & D_2(\tau_m\delta(u_{r_2})) \\ D_2(\tau_1(v_1)) & \cdots & D_2(\tau_m(v_1)) & D_2(\tau_{r'_2+1}\delta(v_1)) & \cdots & D_2(\tau_m\delta(v_1)) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_2(\tau_{r_2}(v_{r_2})) & \cdots & D_2(\tau_m(v_{r_2})) & D_2(\tau_{r'_2+1}\delta(v_{r_2})) & \cdots & D_2(\tau_m\delta(v_{r_2})) \end{pmatrix}$$

has rank  $r_2$ . Note that because the first  $r'_2$  embeddings commute with  $\delta$ , the entries of the last  $r_2$  rows of the first  $r_2$  columns are all 0's. Also, it follows from the equation  $\delta(u_i) = -\delta(u_i)$  and  $\delta(v_i) = \delta(v_i)$ , this matrix has the following block form

$$\begin{pmatrix} A_{r_2 \times r'_2} & B_{r_2 \times (r_2 - r'_2)/2} & -B_{r_2 \times (r_2 - r'_2)/2} \\ 0 & C_{r_2 \times (r_2 - r'_2)/2} & C_{r_2 \times (r_2 - r'_2)/2} \end{pmatrix}$$

So the matrix  $A$  has to have rank  $r'_2$ . For the last  $(r_2 - r'_2)$ -columns to have rank  $r_2 - r'_2$ , the matrices  $B$  and  $C$  must both have maximal possible rank, that is,  $(r_2 - r'_2)/2$ . Since by Borel's Theorem,

$$\text{rank } C = \text{rank } \mathcal{B}_+(F),$$

and  $\text{rank } \mathcal{B}_+(F) + \text{rank } \mathcal{B}_-(F) = r_2$ , Theorem B follows.  $\square$

We can also describe the situation when  $F \subset \mathbb{C}$  is a number field that is not stable under conjugation. If  $E \subset \mathbb{C}$  is any number field containing  $F$  with  $E = \bar{E}$  then  $\mathcal{B}_+(E)$  and  $\mathcal{B}_-(E)$  are defined, so we can form

$$\mathcal{B}_+(F) := \mathcal{B}_+(E) \cap \mathcal{B}(F)_{\mathbb{Q}} \quad \text{and} \quad \mathcal{B}_-(F) := \mathcal{B}_-(E) \cap \mathcal{B}(F)_{\mathbb{Q}}.$$

These subgroups are independent of the choice of  $E$ , but in general they will not sum to  $\mathcal{B}(F)_{\mathbb{Q}}$ .

Denote  $F_{\mathbb{R}} = F \cap \mathbb{R}$  and let  $F' = F \cap \bar{F}$ . Clearly  $F'$  contains  $F_{\mathbb{R}}$ , and  $F_{\mathbb{R}}$  must be the fixed field of conjugation on  $F'$ . Thus either  $F' = F_{\mathbb{R}}$  or  $F'$  is an imaginary quadratic extension of  $F_{\mathbb{R}}$ . Now  $F'$  is a field to which Theorem B applies, so  $\mathcal{B}(F')_{\mathbb{Q}} = \mathcal{B}_+(F') \oplus \mathcal{B}_-(F')$ , with the ranks of the summands given by Theorem B.

**THEOREM 3.1.**  $\mathcal{B}_-(F) = \mathcal{B}_-(F')$  and  $\mathcal{B}_+(F) = \mathcal{B}_+(F') = \mathcal{B}(F_{\mathbb{R}})_{\mathbb{Q}}$ .

**COROLLARY 3.2.**  $\mathcal{B}_+(F)$  is trivial if and only if  $F_{\mathbb{R}}$  is totally real.

$\mathcal{B}_-(F)$  is trivial if and only if  $F' = F_{\mathbb{R}}$ .

$\mathcal{B}_-(F) = \mathcal{B}(F)_{\mathbb{Q}}$  if and only if  $F = F'$  and  $F_{\mathbb{R}}$  is totally real; that is either  $F$  is totally real or the embedding  $F \hookrightarrow \mathbb{C}$  is a CM-embedding.

*Proof of Theorem 3.1.* We work in a Galois superfield  $E$  of  $F$  and identify Bloch groups with their images in  $\mathcal{B}(E)_{\mathbb{Q}}$ . Let  $G = \text{Gal}(E/\mathbb{Q})$ , so  $H = \text{Gal}(E/F) \subset G$  is the subgroup which fixes  $F$ . We fix an embedding  $E \subset \mathbb{C}$  extending the given embedding of  $F$  and denote complex conjugation for this embedding by  $\delta$ . Then the subgroup  $H_{\mathbb{R}}$  generated by  $H$  and  $\delta$  is  $\text{Gal}(E/F_{\mathbb{R}})$ , so it follows from Proposition 2.1 that  $\mathcal{B}_+(F) = \mathcal{B}(E^H)_{\mathbb{Q}} \cap \mathcal{B}(E)_{\mathbb{Q}}^{\delta} = (\mathcal{B}(E)_{\mathbb{Q}}^H)^{\delta} = \mathcal{B}(E)_{\mathbb{Q}}^{H_{\mathbb{R}}} = \mathcal{B}(F_{\mathbb{R}})_{\mathbb{Q}}$ . Moreover,  $\mathcal{B}_-(F)$  is fixed by both  $H$  and  $\delta H \delta$  and hence by the group  $H'$  that they generate. But  $H'$  is the Galois group in  $E$  of  $F \cap \delta(F) = F \cap \bar{F} = F'$ . Thus  $\mathcal{B}_-(F)$  is in

$\mathcal{B}(E)_{\mathbf{Q}}^{H'} = \mathcal{B}(F')_{\mathbf{Q}}$ . We thus obtain inclusions  $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$ , and the reverse inclusions are trivial.  $\square$

*Proof of Corollary 3.2.* The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that  $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$  and if  $F' \neq F$  then  $F$  has strictly more complex embeddings than  $F'$  so  $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$ . Thus, to have  $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$  we must have  $F = F'$ . The claim then follows directly from Theorem B.  $\square$

REMARK. We have pointed out at the beginning of sect. 2 that  $\mathcal{B}(F)$  could have been replaced by  $K_3(F)$  in all our discussions. The analog of Borel's theorem holds for  $K_i(F)$  for all  $i \equiv 3 \pmod{4}$ , so the results described above are also valid for these  $K$ -groups. When  $1 < i \equiv 1 \pmod{4}$  Borel's theorem gives a map  $K_i(F) \rightarrow \mathbf{R}^{r_1+r_2}$  whose kernel is torsion and whose image is a lattice. The only change is that one obtains  $r_1 + \frac{1}{2}(r_2 + r'_2)$  and  $\frac{1}{2}(r_2 - r'_2)$  as the dimensions of the  $+$  and  $-$  eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if  $E \subset \mathbf{C}$  is Galois over  $\mathbf{Q}$  with group  $G$  and  $\delta$  is its conjugation then  $K_i(E) \otimes \mathbf{R}$  is  $G$ -equivariantly isomorphic to  $\{ \sum r_{\gamma} \gamma \in \mathbf{R}G \mid r_{\gamma} = (-1)^{(i-1)/2} r_{\delta\gamma} \}$  for  $i > 1$  and odd.

#### 4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that  $D_2(z)$  represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer  $N \geq 3$ , the real numbers  $D_2(e^{2\pi i \frac{-1+j}{N}})$ , with  $j$  relatively prime to  $N$  and  $0 < j < N/2$ , are linearly independent over the rationals.*

A field homomorphism  $\tau: F \rightarrow K$  clearly induces a homomorphism on the Bloch groups  $\mathcal{B}(F) \rightarrow \mathcal{B}(K)$  which, by abuse of notation, will again be denoted by  $\tau$ .

Given a cyclotomic field  $F = \mathbf{Q}(e^{2\pi i \frac{-1}{N}})$ , the elements  $[e^{2\pi i j/N}]$ , with  $j$  relatively prime to  $N$  and  $0 < j < N/2$ , form a basis of the Bloch group  $\mathcal{B}(F) \otimes \mathbf{Q}$  (see Bloch [2]). Hence Milnor's conjecture can be reformulated that  $D_2: \mathcal{B}(F) \rightarrow \mathbf{R}$  given on generators by  $[z] \mapsto D_2(z)$  is injective for a cyclotomic field  $F$ .