

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HIGHER EULER CHARACTERISTICS (I)
Autor: Geoghegan, Ross / Nicas, Andrew
Kapitel: 5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM
DOI: <https://doi.org/10.5169/seals-61816>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 30.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof. Since $n > 1$, if $T \subset T^n$ is a circle subgroup then $\chi(X/T) = 0$. Applying Theorem 4.5 to the bundle $T \rightarrow X \rightarrow X/T$ yields the conclusion. \square

COROLLARY 4.8. *If $n > 1$ then $\chi_1(T^n): \mathbf{Z}^n \rightarrow \mathbf{Z}^n$ is zero.* \square

5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

Let G be a group of type \mathcal{F} . Gottlieb's theorem (see Propositions 1.3 and 2.4) asserts that if $\chi(G) \neq 0$ then $Z(G)$, the center of G , is trivial. We prove an analogous theorem for $\chi_1(G; \mathbf{Q})$: if $\chi_1(G; \mathbf{Q}) \neq 0$ then the center of G is infinite cyclic provided G satisfies an extra hypothesis (explained below) related to the Bass Conjecture; see Proposition 5.2 and Theorem 5.4.

Throughout this section R will be a commutative ground ring. Let S be any associative R -algebra with unit. The Hochschild homology group $HH_0(S)$ is the R -module $S/[S, S]$ where $[S, S]$ is the R -submodule of S generated by $\{ab - ba \mid a, b \in S\}$; see §2. Recall that $K_0(S)$ is the abelian group F/A where F is the free abelian group generated by the set of all isomorphism classes $[M]$ of finitely generated projective right S -modules $M \subset \bigoplus_{i=1}^{\infty} S$ and A is the subgroup of F generated by relations of the form $[M_1 \oplus M_2] - [M_1] - [M_2]$. Since a finitely generated projective module is the image of a finitely generated free module under an idempotent homomorphism, each element of $K_0(S)$ can be represented by an idempotent matrix over S . The *Hattori-Stallings* trace $T_0: K_0(S) \rightarrow HH_0(S)$ is defined as follows. Let $A: M \rightarrow M$ be an idempotent endomorphism of a free, finitely generated right S -module M representing $x \in K_0(S)$. If $[A]$ is the matrix of A with respect to a given basis for M then $T_0(x)$ is defined to be $T_0([A]) \in HH_0(S)$.

Consider the groupring, RG , of a group G over R . Then $HH_0(RG)$ is naturally isomorphic to the free R -module generated by G_1 , the set of conjugacy classes of G (see §2 for an explanation in the case $R = \mathbf{Z}$). Recall that for $g \in G$ we write $C(g) \in G_1$ for the conjugacy class of g , $HH_0(RG)_{C(g)}$ for the summand of $HH_0(RG)$ corresponding to $C(g)$ and $x_{C(g)}$ for the $C(g)$ -component of $x \in HH_0(RG)$. Also write $HH_0(RG) = HH_0(RG)_{C(1)} \oplus HH_0(RG)'$ where $1 \in G$ is the identity element of G , and $HH_0(RG)'$ is the direct sum of the remaining summands. The augmentation homomorphism $\varepsilon: RG \rightarrow R$ induces a homomorphism $\varepsilon_*: HH_0(RG) \rightarrow HH_0(R) = R$.

STRONG BASS PROPERTY. We say that the group G has the *Strong Bass Property over R* , abbreviated to “SBP over R ”, if the image of the homomorphism $T_0: K_0(RG) \rightarrow HH_0(RG)$ lies in the $HH_0(RG)_{C(1)}$ summand.

WEAK BASS PROPERTY. We say that the group G has the *Weak Bass Property over R* , abbreviated to “WBP over R ”, if the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)' \xrightarrow{\varepsilon_*} R$$

is zero.

Clearly, if G has the SBP over R then it also has WBP over R . There are well-known conjectures concerning the SBP and the WBP (see [Bass], [DV] and [St, §4.1]):

STRONG BASS CONJECTURE. Every group has the SBP over \mathbf{Z} .

WEAK BASS CONJECTURE. Every group has the WBP over \mathbf{Z} .

The corresponding conjectures are false over \mathbf{Q} for a group which has nontrivial torsion; instead, one could conjecture:

STRONG BASS CONJECTURE OVER \mathbf{Q} . Every torsion free group has the SBP over \mathbf{Q} .

WEAK BASS CONJECTURE OVER \mathbf{Q} . Every torsion free group has the WBP over \mathbf{Q} .

Each element of the center of G , $Z(G)$, makes up its own conjugacy class. Given a subgroup N of $Z(G)$, let $HH_0(RG)_N = \bigoplus_{C(g) \in c(N)} HH_0(RG)_{C(g)}$ where $c(N)$ is the set of conjugacy classes in G represented by elements of N . Then $HH_0(RG) = HH_0(RG)_N \oplus HH_0(RG)'_N$ where $HH_0(RG)'_N$ is the direct sum of the summands corresponding to the conjugacy classes not in $c(N)$.

PROPERTY C. We say that the group G has *Property C over R* if there exists a non-empty subset N of $Z(G)$ such that the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)'_N \xrightarrow{\varepsilon_*} R$$

is zero.

By taking N to be the trivial subgroup of $Z(G)$ we see that if G has the WBP over R then it also has Property C over R .

Recall that a group G is said to have finite cohomological dimension over the commutative ground ring R if there exists an integer N such that $H^k(G, M) = 0$ for all RG -modules M and for all $k > N$. Also, G is said to be of type FP_∞ over R if the trivial RG -module R has a resolution by finitely generated projective RG -modules.

The following proposition is derived from the techniques of [St, §3].

PROPOSITION 5.1. *Let R be a principal ideal domain of characteristic $p \geq 0$. Suppose that G is of type FP_∞ over R and has finite cohomological dimension over R . Suppose also that G has a subgroup H of finite index which has Property C over R ; furthermore, if $p > 0$ assume that p does not divide $[G:H]$. If the Euler characteristic $\chi(G; R) \equiv \sum_{i \geq 0} (-1)^i \text{rank}_R H_i(G, R)$ is non-zero modulo p then the center of G is finite.*

Proof. Since H is of finite index in G , H is also of type FP_∞ over R ([Bi, Proposition 2.5]) and has finite cohomological dimension over R ([Bi, Corollary 5.10]). Furthermore, $\chi(H; R) = [G:H] \chi(G; R)$ and so $\chi(H; R) \not\equiv 0 \pmod{p}$.

We show that the center of H , $Z(H)$, is finite. It then follows that the center of G , $Z(G)$, is finite because there is an exact sequence $1 \rightarrow Z(G) \cap H \rightarrow Z(G) \rightarrow N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G , and the groups $N_G(H)/H$ and $Z(G) \cap H \subset Z(H)$ are finite.

Since H is of type FP_∞ over R and has finite cohomological dimension over R , it follows that R has a finite resolution, $0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow R \rightarrow 0$, where each P_j is a finitely generated projective RH -module (combine [Bi, Proposition 4.1 (b)] and [Bi, Proposition 1.5]). Let $\varepsilon: RH \rightarrow R$ be the augmentation homomorphism. Consider the commutative square:

$$\begin{array}{ccc} K_0(RH) & \xrightarrow{T_0} & HH_0(RH) \\ \varepsilon_* \downarrow & & \varepsilon_* \downarrow \\ K_0(R) & \xrightarrow{T_0} & HH_0(R) \cong R \end{array}$$

Let $\alpha = \sum_{n \geq 0} (-1)^n [P_n] \in K_0(RH)$. Then $\varepsilon_*(T_0(\alpha)) = T_0(\varepsilon_*(\alpha)) = \chi(H; R) \cdot 1$ where $1 \in R$ is the unity in R . The second equality is the classical Hopf trace formula over the principal ideal domain R . (Stallings ([St]) calls $T_0(\alpha) \in HH_0(RH)$ the *Euler characteristic* of the projective RH -complex P_* .) Since H is assumed to have Property C over R , there is a non-empty subset N of $Z(H)$ such that $\varepsilon_*(T_0(\alpha)) = \varepsilon_*(T_0(\alpha)_N)$.

Since $\chi(H; R) \neq 0 \pmod p$, it follows that $T_0(\alpha)_{C(h)} \neq 0$ for some $h \in N \subset Z(H)$. Recall that the group $Z(H)$ acts on $HH_0(RH)$ by $(rC(h))\omega = rC(h\omega^{-1})$ where $r \in R$, $h \in H$, and $\omega \in Z(H)$. By [St, Theorem 3.4] (compare (2.3) above), $T_0(\alpha)\omega = T_0(\alpha)$ for all $\omega \in Z(H)$. Since an element of $HH_0(RH)$ is a *finite* linear combination of conjugacy classes, it follows that the condition $T_0(\alpha)_{C(h)} \neq 0$ with h as above is impossible unless $Z(H)$ is finite. \square

We will be interested in groups with the property that certain of their central quotients have Property C “virtually”:

PROPERTY D. Let $p \geq 0$ be the characteristic of R . We say that the group G has *Property D over R* if the following condition holds. Given any element τ in the center of G with the property that the extension class $e_R \in H^2(G/\langle \tau \rangle; R)$ is zero (where $\langle \tau \rangle$ is the cyclic subgroup generated by τ), there is a finite index subgroup $H \subset G/\langle \tau \rangle$ such that H has Property C over R ; moreover, if $p > 0$ we require that p does not divide $[G : H]$.

The next Proposition is our “higher” analog of Gottlieb’s theorem over a field of arbitrary characteristic; Theorem 5.4, below, is a more usable version over \mathbf{Q} .

PROPOSITION 5.2. *Let \mathbf{F} be a field. Suppose G is a group of type \mathcal{F} such that G has Property D over \mathbf{F} . If $\chi_1(G; \mathbf{F}) \neq 0$, then the center of G is infinite cyclic.*

Proof. Let τ be any element in $Z(G)$, the center of G , such that $\chi_1(G; \mathbf{F})(\tau) \neq 0$. Since G is necessarily torsion free, the group $T = \langle \tau \rangle$ is infinite cyclic. By [Bi, Proposition 2.7] G/T is of type FP_∞ over \mathbf{Z} (and hence over any commutative ring). Since T is central, the Serre fibration $S^1 \simeq K(T, 1) \rightarrow K(G, 1) \rightarrow K(G/T, 1)$ is orientable. By Theorem 4.2, $e_{\mathbf{F}} = 0 \in H^2(G/T; \mathbf{F})$, and $\chi(G/T; \mathbf{F})$ exists and is non-zero mod p where $p \geq 0$ is the characteristic of \mathbf{F} . Consider the following portion of the cohomology Gysin sequence of the fibration $S^1 \rightarrow K(G, 1) \rightarrow K(G/T, 1)$, with coefficients in an arbitrary $\mathbf{F}G/T$ -module M :

$$H^{i-2}(G/T; M) \xrightarrow{\cup e_{\mathbf{F}}} H^i(G/T; M) \rightarrow H^i(G; M).$$

Since $e_{\mathbf{F}} = 0$, $H^i(G/T; M) \rightarrow H^i(G; M)$ is injective and so $H^i(G/T, M) = 0$ for $i > \dim X$ where X is a finite complex homotopy equivalent to $K(G, 1)$. In particular, Proposition 5.1 applies to G/T and so the center of G/T is

finite. Since the image of $Z(G)$ in G/T is central, it follows that $Z(G)$ is an extension of T by a finite group. Thus $Z(G)$ is infinite cyclic since G is torsion free. \square

Property D may be hard to verify for an arbitrary coefficient ring R . However, when $R = \mathbf{Q}$ we have:

PROPOSITION 5.3. *Let G be a finitely generated group which has the WBP over \mathbf{Q} . Then G has Property D over \mathbf{Q} .*

Proof. Suppose $\tau \in Z(G)$ is such that the extension class $e_{\mathbf{Q}} \in H^2(G/T; \mathbf{Q})$ is zero where T is the cyclic subgroup of G generated by τ . Consider the following portion of the long exact sequence in cohomology associated to the short exact sequence of coefficients, $0 \rightarrow \mathbf{Z} \xrightarrow{j} \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$:

$$H^1(G/T; \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta} H^2(G/T; \mathbf{Z}) \xrightarrow{j_*} H^2(G/T; \mathbf{Q}) .$$

By exactness, $j_*(e_{\mathbf{Z}}) = e_{\mathbf{Q}} = 0$ implies $e_{\mathbf{Z}} = \delta(u)$ for some $u \in H^1(G/T, \mathbf{Q}/\mathbf{Z})$. Let $H = \ker(u)$ where we regard u as an element of $\text{Hom}(G/T, \mathbf{Q}/\mathbf{Z}) \cong H^1(G/T, \mathbf{Q}/\mathbf{Z})$. Since G is finitely generated, $H \hookrightarrow G/T$ is of finite index. Let $H' = \pi^{-1}(H)$ where $\pi: G \rightarrow G/T$ is the quotient homomorphism. Then H' is isomorphic to $H \times T$ because $i^*(e_{\mathbf{Z}}) = 0$. In particular, H is isomorphic to a subgroup of G . Let $\mu: H \rightarrow G$ be a monomorphism. The commutative diagram

$$\begin{array}{ccc} K_0(\mathbf{Q}H) & \xrightarrow{T_0} & HH_0(\mathbf{Q}H) \\ \mu_* \downarrow & & \mu_* \downarrow \\ K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G) \end{array}$$

and the observation that $\mu_*(HH_0(\mathbf{Q}H))_{C(1)} \subset HH_0(\mathbf{Q}G)_{C(1)}$ and $\mu_*(HH_0(\mathbf{Q}H)') \subset HH_0(\mathbf{Q}G)'$ imply that H has the WBP over \mathbf{Q} (and thus Property C over \mathbf{Q}). \square

Combining Propositions 5.2 and 5.3 we get:

THEOREM 5.4. *Suppose that G is a group of type \mathcal{F} and has the WBP over \mathbf{Q} . If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic.* \square

Groups of type \mathcal{F} are a very special class of torsion free groups; one would hope that all groups of type \mathcal{F} have the WBP over \mathbf{Q} . There are special classes of groups of type \mathcal{F} which are known to have the WBP over \mathbf{Q} . We recall two such classes.

A group G is a *linear group* if it is a subgroup of $GL(n, \mathbf{K})$ where \mathbf{K} is a field of characteristic zero. Bass [Bass, Theorem 9.6] proved that a torsion free linear group has the SBP over \mathbf{C} (and thus has the WBP over \mathbf{Q}); also see [Eck].

COROLLARY 5.5. *Suppose G is a linear group of type \mathcal{F} . If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic. \square*

Eckmann [Eck] proved that a group of cohomological dimension 2 over \mathbf{Q} has the SBP over \mathbf{Q} . Consequently:

COROLLARY 5.6. *Suppose G is of type \mathcal{F} and has cohomological dimension 2 over \mathbf{Q} . If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic. \square*

There is a sense in which we can say that $\chi_1(G; \mathbf{Q})$ is an integer. Denote the composite homomorphism $Z(G) \hookrightarrow G \xrightarrow{A} H_1(G; \mathbf{Z}) \rightarrow H_1(G; \mathbf{Q})$ by $A_{\mathbf{Q}}: Z(G) \rightarrow H_1(G; \mathbf{Q})$.

THEOREM 5.7. *Let G be a group of type \mathcal{F} which has the WBP over \mathbf{Q} . Then there exists an integer n_G (depending only on G) such that $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$.*

Proof. If $\chi_1(G; \mathbf{Q}) = 0$ take $n_G = 0$. If $\chi_1(G; \mathbf{Q}) \neq 0$ then by Theorem 5.4 the center of G is infinite cyclic. Let $\tau \in Z(G)$ generate $Z(G)$. Since $\chi_1(G; \mathbf{Q}) \neq 0$ we have $\chi_1(G; \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, $\chi_1(G; \mathbf{Q})(\tau) = -\chi(G/\langle \tau \rangle; \mathbf{Q})\{\tau\}$. Then for any integer r : $\chi_1(G; \mathbf{Q})(\tau^r) = r\chi_1(G; \mathbf{Q})(\tau) = -r\chi(G/\langle \tau \rangle; \mathbf{Q})\{\tau\} = -\chi(G/\langle \tau \rangle; \mathbf{Q})A_{\mathbf{Q}}(\tau^r)$. Thus $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$ with $n_G = -\chi(G/\langle \tau \rangle; \mathbf{Q})$. \square

Remarks.

1. All integers occur as n_G for some G . Given $n \in \mathbf{Z}$, there is a group H of type \mathcal{F} with $\chi(H) = -n$ (e.g. take H to be an appropriate Cartesian product of free groups). Let $G = H \times T$ where T is infinite cyclic. Clearly, $\chi(G/\langle \tau \rangle; \mathbf{Q}) = \chi(H)$ where τ is a generator of $(1) \times T \subset G$ and so $\chi_1(G; \mathbf{Q}) = nA_{\mathbf{Q}}$ (alternatively, see Example 6.15).

2. Theorem 5.7 remains true without the hypothesis that G has the WBP over \mathbf{Q} although the proof is considerably more lengthy. To prove this strengthened result, one shows that for *any* group G of type \mathcal{F} :

- (a) The restriction of $\chi_1(G; \mathbf{Q})$ to $Z(G) \cap [G, G]$ is zero.
 (b) If $\chi_1(G; \mathbf{Q}) \neq 0$ then $\dim_{\mathbf{Q}} A_{\mathbf{Q}}(Z(G)) = 1$.

The desired conclusion follows easily from (a), (b) and Theorem 4.2.

Theorem 5.7 raises the question: For what groups G of type \mathcal{F} is $\chi_1(G, \mathbf{Q}) \neq 0$? We give a necessary condition. Recall that a group H has type \mathcal{FD} if there is a finitely dominated $K(H, 1)$ (i.e. $K(H, 1)$ is a homotopy retract of a finite complex).

PROPOSITION 5.8. *If $\chi_1(G, \mathbf{Q}) \neq 0$ then G is isomorphic to a semidirect product $\langle H, t \mid tht^{-1} = \theta(h) \text{ for all } h \in H \rangle$ where H has type \mathcal{FD} .*

Proof. Let $\tau \in Z(G)$ be such that $\chi_1(G, \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, it follows that $\{\tau\} \in H_1(G) \cong G_{\text{ab}}$ is of infinite order. Thus there is an epimorphism $p: G \rightarrow \mathbf{Z}$ with $p(\tau) = n$ for some $n > 0$. Let $H = \ker(p)$. Since $\tau \in Z(G)$, $p^{-1}(n\mathbf{Z}) \cong H \times \mathbf{Z}$ and has finite index in G . Thus $H \times \mathbf{Z}$ has type \mathcal{F} and so H has type \mathcal{FD} . \square

Thus it is worthwhile to compute $\chi_1(G, \mathbf{Q})$ in terms of such a semidirect product structure. The geometric problem underlying this is the study of $\chi_1(X)$ where X is a mapping torus. We study this next, returning to the group theoretic case in §7.

6. MAPPING TORI

In this section, we consider $\chi_1(X)$ and $\tilde{\chi}_1(X)$ when X is the mapping torus of a map $f: Z \rightarrow Z$. The main results are Theorems 6.3, 6.13, 6.14, 6.16 and Corollary 6.18. Applications to the aspherical case will be given in §7.

Suppose Z is a path connected space and has a basepoint $v \in Z$. Given a continuous map $f: Z \rightarrow Z$, its *mapping torus*, denoted by $T(Z, f)$, is the space obtained from $Z \times [0, 1]$ by identifying $(z, 1)$ with $(f(z), 0)$ for each $z \in Z$. The image of $(z, u) \in Z \times [0, 1]$ in $T(Z, f)$ will be denoted by $[z, u]$. Choose a basepath σ from v to $f(v)$ and let $\theta: H \rightarrow H$ be the self homomorphism of $H \equiv \pi_1(Z, v)$ determined by f and σ .

Let $X = T(Z, f)$. Choose $w = [v, 0]$ as a basepoint for X and let $G = \pi_1(X, w)$. There is a canonical map of X to the standard circle S^1 (realized as complex numbers of unit modulus) given by: $p_f: X \rightarrow S^1$, $p_f([z, s]) = e^{2\pi is}$. Let $i: Z \hookrightarrow X$ be the inclusion $z \mapsto [z, 0]$.