Proof. Since $n > 1$, if $T \subset T^n$ is a circle subgroup then $\chi(X/T) = 0$. Applying Theorem 4.5 to the bundle $T \to X \to X/T$ yields the conclusion. $\square$

COROLLARY 4.8. *If* $n > 1$ *then* $\chi_1(T^n): \mathbf{Z}^n \to \mathbf{Z}^n$ *is zero.* $\square$

5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

Let G be a group of type $\mathscr{F}$. Gottlieb's theorem (see Propositions 1.3 and 2.4) asserts that if $\chi(G) \neq 0$ then $Z(G)$, the center of G, is trivial. We prove an analogous theorem for $\chi_1(G; \mathbf{Q})$: if $\chi_1(G; \mathbf{Q}) \neq 0$ then the center of G is infinite cyclic provided G satisfies an extra hypothesis (explained below) related to the Bass Conjecture; see Proposition 5.2 and Theorem 5.4.

Throughout this section R will be a commutative ground ring. Let S be any associative R-algebra with unit. The Hochschild homology group $HH_0(S)$ is the R-module $S/[S, S]$ where $[S, S]$ is the R-submodule of S generated by $\{ab - ba \mid a, b \in S\}$; see §2. Recall that $K_0(S)$ is the abelian group F/A where F is the free abelian group generated by the set of all isomorphism classes $[M]$ of finitely generated projective right S-modules $M \subset \oplus_{i=1}^{\infty} S$ and A is the subgroup of F generated by relations of the form $[M_1 \oplus M_2] - [M_1] - [M_2]$. Since a finitely generated projective module is the image of a finitely generated free module under an idempotent homomorphism, each element of $K_0(S)$ can be represented by an idempotent matrix over S. The *Hattori-Stallings* trace $T_0: K_0(S) \to HH_0(S)$ is defined as follows. Let $A: M \to M$ be an idempotent endomorphism of a free, finitely generated right S-module M representing $x \in K_0(S)$. If $[A]$ is the matrix of A with respect to a given basis for M then $T_0(x)$ is defined to be $T_0([A]) \in HH_0(S)$.

Consider the groupring, RG, of a group G over R. Then $HH_0(RG)$ is naturally isomorphic to the free R-module generated by G_1, the set of conjugacy classes of G (see §2 for an explanation in the case $R = \mathbf{Z}$). Recall that for $g \in G$ we write $C(g) \in G_1$ for the conjugacy class of g, $HH_0(RG)_{C(g)}$ for the summand of $HH_0(RG)$ corresponding to $C(g)$ and $x_{C(g)}$ for the $C(g)$-component of $x \in HH_0(RG)$. Also write $HH_0(RG) = HH_0(RG)_{C(1)} \oplus HH_0(RG)'$ where $1 \in G$ is the identity element of G, and $HH_0(RG)'$ is the direct sum of the remaining summands. The augmentation homomorphism $\varepsilon: RG \to R$ induces a homomorphism $\varepsilon_*: HH_0(RG) \to HH_0(R) = R$.

STRONG BASS PROPERTY. We say that the group G has the *Strong Bass Property over R*, abbreviated to "SBP over R", if the image of the homomorphism $T_0 : K_0(RG) \to HH_0(RG)$ lies in the $HH_0(RG)_{C(1)}$ summand.

WEAK BASS PROPERTY. We say that the group G has the *Weak Bass Property over R*, abbreviated to "WBP over R", if the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)' \xrightarrow{\varepsilon_*} R$$

is zero.

Clearly, if G has the SBP over R then it also has WBP over R. There are well-known conjectures concerning the SBP and the WBP (see [Bass], [DV] and [St, §4.1]):

STRONG BASS CONJECTURE. Every group has the SBP over $\mathbf{Z}$.

WEAK BASS CONJECTURE. Every group has the WBP over $\mathbf{Z}$.

The corresponding conjectures are false over $\mathbf{Q}$ for a group which has nontrivial torsion; instead, one could conjecture:

STRONG BASS CONJECTURE OVER $\mathbf{Q}$. Every torsion free group has the SBP over $\mathbf{Q}$.

WEAK BASS CONJECTURE OVER $\mathbf{Q}$. Every torsion free group has the WBP over $\mathbf{Q}$.

Each element of the center of G, $Z(G)$, makes up its own conjugacy class. Given a subgroup N of $Z(G)$, let $HH_0(RG)_N = \bigoplus_{C(g) \in c(N)} HH_0(RG)_{C(g)}$ where $c(N)$ is the set of conjugacy classes in G represented by elements of N. Then $HH_0(RG) = HH_0(RG)_N \oplus HH_0(RG)'_N$ where $HH_0(RG)'_N$ is the direct sum of the summands corresponding to the conjugacy classes not in $c(N)$.

PROPERTY C. We say that the group G has *Property C over R* if there exists a non-empty subset N of $Z(G)$ such that the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)'_N \xrightarrow{\varepsilon_*} R$$

is zero.

By taking N to be the trivial subgroup of $Z(G)$ we see that if G has the WBP over R then it also has Property C over R.

Recall that a group G is said to have finite cohomological dimension over the commutative ground ring R if there exists an integer N such that $H^k(G, M) = 0$ for all RG-modules M and for all $k > N$. Also, G is said to be of type FP_∞ over R if the trivial RG-module R has a resolution by finitely generated projective RG-modules.

The following proposition is derived from the techniques of [St, §3].

PROPOSITION 5.1. *Let R be a principal ideal domain of characteristic $p \geqslant 0$. Suppose that G is of type FP_∞ over R and has finite cohomological dimension over R. Suppose also that G has a subgroup H of finite index which has Property C over R; furthermore, if $p > 0$ assume that p does not divide $[G:H]$. If the Euler characteristic $\chi(G; R) \equiv \sum_{i \geqslant 0}(-1)^i \operatorname{rank}_R H_i(G, R)$ is non-zero modulo p then the center of G is finite.*

Proof. Since H is of finite index in G, H is also of type FP_∞ over R ([Bi, Proposition 2.5]) and has finite cohomological dimension over R ([Bi, Corollary 5.10]). Furthermore, $\chi(H; R) = [G:H]\,\chi(G; R)$ and so $\chi(H; R) \neq 0 \bmod p$.

We show that the center of H, $Z(H)$, is finite. It then follows that the center of G, $Z(G)$, is finite because there is an exact sequence $1 \to Z(G) \cap H \to Z(G) \to N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G, and the groups $N_G(H)/H$ and $Z(G) \cap H \subset Z(H)$ are finite.

Since H is of type FP_∞ over R and has finite cohomological dimension over R, it follows that R has a finite resolution, $0 \to P_n \to \cdots \to P_0 \to R \to 0$, where each P_j is a finitely generated projective RH-module (combine [Bi, Proposition 4.1(b)] and [Bi, Proposition 1.5])). Let $\varepsilon: RH \to R$ be the augmentation homomorphism. Consider the commutative square:

$$\begin{array}{ccc} K_0(RH) & \xrightarrow{T_0} & HH_0(RH) \\ \varepsilon_* \downarrow & & \varepsilon_* \downarrow \\ K_0(R) & \xrightarrow{T_0} & HH_0(R) \cong R \end{array}$$

Let $\alpha = \sum_{n \geqslant 0}(-1)^n [P_n] \in K_0(RH)$. Then $\varepsilon_*(T_0(\alpha)) = T_0(\varepsilon_*(\alpha)) = \chi(H; R) \cdot 1$ where $1 \in R$ is the unity in R. The second equality is the classical Hopf trace formula over the principal ideal domain R. (Stallings ([St]) calls $T_0(\alpha) \in HH_0(RH)$ the *Euler characteristic* of the projective RH-complex P_*.) Since H is assumed to have Property C over R, there is a non-empty subset N of $Z(H)$ such that $\varepsilon_*(T_0(\alpha)) = \varepsilon_*(T_0(\alpha)_N)$.

Since $\chi(H;R) \neq 0 \mod p$, it follows that $T_0(\alpha)_{C(h)} \neq 0$ for some $h \in N \subset Z(H)$. Recall that the group $Z(H)$ acts on $HH_0(RH)$ by $(rC(h))\omega = rC(h\omega^{-1})$ where $r \in R$, $h \in H$, and $\omega \in Z(H)$. By [St, Theorem 3.4] (compare (2.3) above), $T_0(\alpha)\omega = T_0(\alpha)$ for all $\omega \in Z(H)$. Since an element of $HH_0(RH)$ is a *finite* linear combination of conjugacy classes, it follows that the condition $T_0(\alpha)_{C(h)} \neq 0$ with h as above is impossible unless $Z(H)$ is finite. □

We will be interested in groups with the property that certain of their central quotients have Property C "virtually":

PROPERTY D. Let $p \geqslant 0$ be the characteristic of R. We say that the group G has *Property D over* R if the following condition holds. Given any element τ in the center of G with the property that the extension class $e_R \in H^2(G/\langle\tau\rangle; R)$ is zero (where $\langle\tau\rangle$ is the cyclic subgroup generated by τ), there is a finite index subgroup $H \subset G/\langle\tau\rangle$ such that H has Property C over R; moreover, if $p > 0$ we require that p does not divide $[G:H]$.

The next Proposition is our "higher" analog of Gottlieb's theorem over a field of arbitrary characteristic; Theorem 5.4, below, is a more usable version over $\mathbf{Q}$.

PROPOSITION 5.2. *Let* $\mathbf{F}$ *be a field. Suppose* G *is a group of type* $\mathcal{F}$ *such that* G *has Property D over* $\mathbf{F}$. *If* $\chi_1(G;\mathbf{F}) \neq 0$, *then the center of* G *is infinite cyclic.*

Proof. Let τ be any element in $Z(G)$, the center of G, such that $\chi_1(G;\mathbf{F})(\tau) \neq 0$. Since G is necessarily torsion free, the group $T = \langle\tau\rangle$ is infinite cyclic. By [Bi, Proposition 2.7] G/T is of type FP_∞ over $\mathbf{Z}$ (and hence over any commutative ring). Since T is central, the Serre fibration $S^1 \simeq K(T,1) \to K(G,1) \to K(G/T,1)$ is orientable. By Theorem 4.2, $e_{\mathbf{F}} = 0 \in H^2(G/T;\mathbf{F})$, and $\chi(G/T;\mathbf{F})$ exists and is non-zero mod p where $p \geqslant 0$ is the characteristic of $\mathbf{F}$. Consider the following portion of the cohomology Gysin sequence of the fibration $S^1 \to K(G,1) \to K(G/T,1)$, with coefficients in an arbitrary $\mathbf{F}G/T$-module M:

$$H^{i-2}(G/T;M) \xrightarrow{\cup e_{\mathbf{F}}} H^i(G/T;M) \to H^i(G;M)\,.$$

Since $e_{\mathbf{F}} = 0$, $H^i(G/T;M) \to H^i(G;M)$ is injective and so $H^i(G/T,M) = 0$ for $i > \dim X$ where X is a finite complex homotopy equivalent to $K(G,1)$. In particular, Proposition 5.1 applies to G/T and so the center of G/T is finite. Since the image of $Z(G)$ in G/T is central, it follows that $Z(G)$ is an extension of T by a finite group. Thus $Z(G)$ is infinite cyclic since G is torsion free. □

Property D may be hard to verify for an arbitrary coefficient ring R. However, when $R = \mathbf{Q}$ we have:

PROPOSITION 5.3. *Let* G *be a finitely generated group which has the WBP over* $\mathbf{Q}$. *Then* G *has Property D over* $\mathbf{Q}$.

Proof. Suppose $\tau \in Z(G)$ is such that the extension class $e_{\mathbf{Q}} \in H^2(G/T; \mathbf{Q})$ is zero where T is the cyclic subgroup of G generated by τ. Consider the following portion of the long exact sequence in cohomology associated to the short exact sequence of coefficients, $0 \to \mathbf{Z} \xrightarrow{j} \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$:

$$H^1(G/T; \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta} H^2(G/T; \mathbf{Z}) \xrightarrow{j_*} H^2(G/T; \mathbf{Q}) .$$

By exactness, $j_*(e_{\mathbf{Z}}) = e_{\mathbf{Q}} = 0$ implies $e_{\mathbf{Z}} = \delta(u)$ for some $u \in H^1(G/T, \mathbf{Q}/\mathbf{Z})$. Let $H = \ker(u)$ where we regard u as an element of $\mathrm{Hom}(G/T, \mathbf{Q}/\mathbf{Z}) \cong H^1(G/T, \mathbf{Q}/\mathbf{Z})$. Since G is finitely generated, $H \overset{i}{\hookrightarrow} G/T$ is of finite index. Let $H' = \pi^{-1}(H)$ where $\pi : G \to G/T$ is the quotient homomorphism. Then H' is isomorphic to $H \times T$ because $i^*(e_{\mathbf{Z}}) = 0$. In particular, H is isomorphic to a subgroup of G. Let $\mu : H \to G$ be a monomorphism. The commutative diagram

$$\begin{array}{ccc} K_0(\mathbf{Q}H) & \xrightarrow{T_0} & HH_0(\mathbf{Q}H) \\ \mu_* \downarrow & & \mu_* \downarrow \\ K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G) \end{array}$$

and the observation that $\mu_*(HH_0(\mathbf{Q}H))_{C(1)} \subset HH_0(\mathbf{Q}G)_{C(1)}$ and $\mu_*(HH_0(\mathbf{Q}H)') \subset HH_0(\mathbf{Q}G)'$ imply that H has the WBP over $\mathbf{Q}$ (and thus Property C over $\mathbf{Q}$). □

Combining Propositions 5.2 and 5.3 we get:

THEOREM 5.4. *Suppose that* G *is a group of type* $\mathscr{F}$ *and has the WBP over* $\mathbf{Q}$. *If* $\chi_1(G; \mathbf{Q}) \neq 0$, *then the center of* G *is infinite cyclic.* □

Groups of type $\mathscr{F}$ are a very special class of torsion free groups; one would hope that all groups of type $\mathscr{F}$ have the WBP over $\mathbf{Q}$. There are special classes of groups of type $\mathscr{F}$ which are known to have the WBP over $\mathbf{Q}$. We recall two such classes.

A group G is a *linear group* if it is a subgroup of $GL(n, \mathbf{K})$ where $\mathbf{K}$ is a field of characteristic zero. Bass [Bass, Theorem 9.6] proved that a torsion free linear group has the SBP over $\mathbf{C}$ (and thus has the WBP over $\mathbf{Q}$); also see [Eck].

COROLLARY 5.5. *Suppose G is a linear group of type $\mathscr{F}$. If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic.* $\square$

Eckmann [Eck] proved that a group of cohomological dimension 2 over $\mathbf{Q}$ has the SBP over $\mathbf{Q}$. Consequently:

COROLLARY 5.6. *Suppose G is of type $\mathscr{F}$ and has cohomological dimension 2 over $\mathbf{Q}$. If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic.* $\square$

There is a sense in which we can say that $\chi_1(G; \mathbf{Q})$ is an integer. Denote the composite homomorphism $Z(G) \hookrightarrow G \xrightarrow{A} H_1(G; \mathbf{Z}) \to H_1(G; \mathbf{Q})$ by $A_{\mathbf{Q}}: Z(G) \to H_1(G; \mathbf{Q})$.

THEOREM 5.7. *Let G be a group of type $\mathscr{F}$ which has the WBP over $\mathbf{Q}$. Then there exists an integer n_G (depending only on G) such that $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$.*

Proof. If $\chi_1(G; \mathbf{Q}) = 0$ take $n_G = 0$. If $\chi_1(G; \mathbf{Q}) \neq 0$ then by Theorem 5.4 the center of G is infinite cyclic. Let $\tau \in Z(G)$ generate $Z(G)$. Since $\chi_1(G; \mathbf{Q}) \neq 0$ we have $\chi_1(G; \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, $\chi_1(G; \mathbf{Q})(\tau) = -\chi(G/\langle\tau\rangle; \mathbf{Q})\{\tau\}$. Then for any integer r: $\chi_1(G; \mathbf{Q})(\tau^r) = r\chi_1(G; \mathbf{Q})(\tau) = -r\chi(G/\langle\tau\rangle; \mathbf{Q})\{\tau\} = -\chi(G/\langle\tau\rangle; \mathbf{Q}) A_{\mathbf{Q}}(\tau^r)$. Thus $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$ with $n_G = -\chi(G/\langle\tau\rangle; \mathbf{Q})$. $\square$

Remarks.

1. All integers occur as n_G for some G. Given $n \in \mathbf{Z}$, there is a group H of type $\mathscr{F}$ with $\chi(H) = -n$ (e.g. take H to be an appropriate Cartesian product of free groups). Let $G = H \times T$ where T is infinite cyclic. Clearly, $\chi(G/\langle\tau\rangle; \mathbf{Q}) = \chi(H)$ where τ is a generator of $(1) \times T \subset G$ and so $\chi_1(G; \mathbf{Q}) = nA_{\mathbf{Q}}$ (alternatively, see Example 6.15).

2. Theorem 5.7 remains true without the hypothesis that G has the WBP over $\mathbf{Q}$ although the proof is considerably more lengthy. To prove this strengthened result, one shows that for *any* group G of type $\mathscr{F}$:

(a) The restriction of $\chi_1(G; \mathbf{Q})$ to $Z(G) \cap [G, G]$ is zero.

(b) If $\chi_1(G; \mathbf{Q}) \neq 0$ then $\dim_{\mathbf{Q}} A_{\mathbf{Q}}(Z(G)) = 1$.

The desired conclusion follows easily from (a), (b) and Theorem 4.2.

Theorem 5.7 raises the question: For what groups G of type $\mathscr{F}$ is $\chi_1(G, \mathbf{Q}) \neq 0$? We give a necessary condition. Recall that a group H has type $\mathscr{F}\mathscr{D}$ if there is a finitely dominated $K(H, 1)$ (i.e. $K(H, 1)$ is a homotopy retract of a finite complex).

PROPOSITION 5.8. *If* $\chi_1(G, \mathbf{Q}) \neq 0$ *then* G *is isomorphic to a semidirect product* $\langle H, t \mid tht^{-1} = \theta(h)$ *for all* $h \in H \rangle$ *where* H *has type* $\mathscr{F}\mathscr{D}$.

Proof. Let $\tau \in Z(G)$ be such that $\chi_1(G, \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, it follows that $\{\tau\} \in H_1(G) \equiv G_{\mathrm{ab}}$ is of infinite order. Thus there is an epimorphism $p: G \to \mathbf{Z}$ with $p(\tau) = n$ for some $n > 0$. Let $H = \ker(p)$. Since $\tau \in Z(G)$, $p^{-1}(n\mathbf{Z}) \cong H \times \mathbf{Z}$ and has finite index in G. Thus $H \times \mathbf{Z}$ has type $\mathscr{F}$ and so H has type $\mathscr{F}\mathscr{D}$. $\square$

Thus it is worthwhile to compute $\chi_1(G, \mathbf{Q})$ in terms of such a semidirect product structure. The geometric problem underlying this is the study of $\chi_1(X)$ where X is a mapping torus. We study this next, returning to the group theoretic case in §7.

6. MAPPING TORI

In this section, we consider $\chi_1(X)$ and $\tilde{\chi}_1(X)$ when X is the mapping torus of a map $f: Z \to Z$. The main results are Theorems 6.3, 6.13, 6.14, 6.16 and Corollary 6.18. Applications to the aspherical case will be given in §7.

Suppose Z is a path connected space and has a basepoint $v \in Z$. Given a continuous map $f: Z \to Z$, its *mapping torus*, denoted by $T(Z, f)$, is the space obtained from $Z \times [0, 1]$ by identifying $(z, 1)$ with $(f(z), 0)$ for each $z \in Z$. The image of $(z, u) \in Z \times [0, 1]$ in $T(Z, f)$ will be denoted by $[z, u]$. Choose a basepath σ from v to $f(v)$ and let $\theta: H \to H$ be the self homomorphism of $H \equiv \pi_1(Z, v)$ determined by f and σ.

Let $X = T(Z, f)$. Choose $w = [v, 0]$ as a basepoint for X and let $G = \pi_1(X, w)$. There is a canonical map of X to the standard circle S^1 (realized as complex numbers of unit modulus) given by: $p_f: X \to S^1$, $p_f([z, s]) = e^{2\pi i s}$. Let $i: Z \hookrightarrow X$ be the inclusion $z \mapsto [z, 0]$.