7. Some applications to hypercomplex function theory

The *Clifford algebra* associated with $\mathbf{R}^n$ endowed with the Euclidean metric is the enlargement of $\mathbf{R}^n$ to a unitary algebra $\mathscr{A}_n$ not generated (as an algebra) by any proper subspace of $\mathbf{R}^n$ and such that $x^2 = -|x|^2$, for any $x \in \mathbf{R}^n$. By polarization, this identity becomes

$$xy + yx = -2\langle x, y\rangle ,$$

for any $x, y \in \mathbf{R}^n$. In particular, if $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbf{R}^n$, one should have

$$e_j e_k + e_k e_j = -2\delta_{jk} .$$

Consequently, $e_j^2 = -1$ and $e_j e_k = -e_k e_j$ for any $j \neq k$. In particular, any element $u \in \mathscr{A}_n$ can be uniquely represented in the form $u = \sum_{k=0}^n \sum'_{|I|=k} u_I e_I$, with $u_I \in \mathbf{R}$, where e_I stands for the product $e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k}$ if $I = (i_1, i_2, \ldots, i_k)$ (we make the convention that $e_\varnothing := 1$). More detailed accounts on these matters can be found in [BDS], [Mi].

The higher dimensional analogue of the form dz extensively used in the complex analysis of one variable is the $\mathscr{A}_n$-valued $(n-1)$-form

$$\omega := \sum_{j=1}^n (-1)^{j-1} e_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n .$$

For a compact Lipschitz domain Ω in $\mathbf{R}^n$, we let $d\sigma$ stand for the usual surface measure induced on $\partial\Omega$ by the Euclidean metric on $\mathbf{R}^n$, and let N denote the outward unit normal to Ω defined $d\sigma$-almost everywhere on $\partial\Omega$. As $\mathbf{R}^n \subset \mathscr{A}_n$, the vector valued function N can also be regarded as a $\mathscr{A}_n$-valued function on $\partial\Omega$. In fact, if ι denotes the inclusion of $\partial\Omega$ into $\mathbf{R}^n$, then

$$\iota^*(\omega) = N d\sigma .$$

An $\mathscr{A}_n$-valued function u defined on an open subset Ω of $\mathbf{R}^n$ is called *integrally continuous,* etc, provided the $\mathscr{A}_n$-valued $(n-1)$-form $u\omega$ has the corresponding property. Recall the generalized Cauchy-Riemann operator

$$D := \sum_{j=1}^n e_j \partial_j .$$

Let $\mathscr{R}(\Omega)$ be as defined at the beginning of §6. We also make the following definition.

DEFINITION 7.1.

(1) If $u = \sum_I' u_I e_I$ *is an* $\mathscr{A}_n$*-valued function defined on* $\Omega \subseteq \mathbf{R}^n$ *whose components* $(u_I)_I$ *are differentiable functions at a point* $a \in \Omega$, *then we define the action of* D *on* u *at* $a \in \Omega$ *by*

$$(Du)(a) := \sum_{i=1}^{n} \sum_{I}' \frac{\partial u_I}{\partial x_i}(a) e_i e_I .$$

(2) If u *and* f *are two locally integrable* $\mathscr{A}_n$*-valued functions on* Ω, *then we say that* $Du = f$ *in the distribution sense on* Ω *provided*

$$\iint_\Omega (D\psi) u\, dx = -\iint_\Omega \psi f\, dx$$

for any real-valued, smooth functions ψ, *compactly supported in* Ω.

(3) A locally $(n-1)$*-integrable,* $\mathscr{A}_n$*-valued function* u *is called Clifford differentiable at* $a \in \Omega$ *if the limit*

$$u'(a) := \lim_{Q \downarrow a} \frac{1}{\lambda_n(Q)} \int_{\partial Q} N u\, d\sigma$$

exists in $\mathscr{A}_n$.

The solutions of the (generalized) Cauchy-Riemann equations $Du = 0$ are called *monogenic functions.*

The theorems we are about to describe now are more or less immediate corollaries of the results obtained so far and we shall omit the proofs.

THEOREM 7.2. *Let* u *be a integrally continuous* $\mathscr{A}_n$*-valued function on the open set* Ω *of* $\mathbf{R}^n \subset \mathscr{A}_n$. *The following are equivalent.*

(1) There exists $f \in L^1_{\mathrm{loc}}(\Omega, \mathscr{A}_n)$ *such that*

$$\int_{\partial Q} N u\, d\sigma = \iint_Q f\, dx$$

for any $Q \in \mathscr{R}(\Omega)$.

(2) There exists a real-valued, positive function $g \in L^1(\Omega, \mathrm{loc})$ *such that*

$$\left| \int_{\partial Q} N u\, d\sigma \right| \leqslant \iint_Q g\, dx$$

for any $Q \in \mathscr{R}(\Omega)$.

(3) For any $Q \in \mathscr{R}(\Omega)$ *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\sum_{i \in J} \left| \int_{\partial Q_i} N_i u \, d\sigma_i \right| \leqslant \varepsilon ,$$

for any subdivision $(Q_i)_{i \in I}$ *of* Q *and any* $J \subseteq I$ *such that* $\sum_{i \in J} \lambda_n(Q_i) \leqslant \delta$.

(4) The function u *is Clifford differentiable almost everywhere on* Ω, u' *is locally integrable on* Ω *and*

$$\int_{\partial Q} N u \, d\sigma = \iint_Q u' \, dx$$

for any $Q \in \mathscr{R}(\Omega)$.

(5) The function u *is Clifford differentiable almost everywhere on* Ω, u' *is locally integrable on* Ω *and*

$$\int_{\partial K} N u \, d\sigma = \iint_{\mathring{K}} u' \, dx$$

for any compact Lipschitz domain $K \subset \Omega$.

(6) Du, *taken in the distribution sense, belongs to* $L^1_{\mathrm{loc}}(\Omega, \mathscr{A}_n)$.

If these equivalent conditions are fulfilled, then also $u' = Du$ *a.e. on* Ω.

THEOREM 7.3. *Let* u *be a* $\mathscr{A}_n$*-valued, uniformly* $(n-1)$*-integrable function in* $\mathbf{R}^n$, *which is absolutely continuous in the special Lipschitz domain* Ω *of* $\mathbf{R}^n \subset \mathscr{A}_n$. *Also, suppose that* $\operatorname{supp} u$ *is compact.*

$$\mu_{n-1}(\operatorname{supp} u \cap \bar{\Omega} \setminus \Omega) = 0 ,$$

u *is integrable on* $b\Omega$, *and that* Du *is integrable on* Ω.
Then

$$\int_{b\Omega} N u \, d\sigma = \iint_{\mathring{\Omega}} Du \, dx .$$

The next application is a refined version of the Pompeiu integral representation formula for $\mathscr{A}_n$-valued functions ([Mo], [Te]). To this effect, we shall call a locally $(n-1)$-integrable function u *mean-continuous at* $a \in \Omega$ if

$$\lim_{Q \downarrow a} \frac{1}{\mu_{n-1}(Q)} \int_{\partial Q} |u(x) - u(a)| \, d\sigma = 0 .$$

Also, let ω_n stand for the area of the unit sphere in $\mathbf{R}^n$.

THEOREM 7.4. *Let* Ω *be a compact Lipschitz domain in* $\mathbf{R}^n \subset \mathscr{A}_n$ *and let* u *be a* $\mathscr{A}_n$*-valued, uniformly* $(n-1)$*-locally integrable function on* $\mathbf{R}^n$, *which is absolutely continuous on* Ω *and mean-continuous almost everywhere on* Ω. *Then, at almost every point* $a \in \mathring{\Omega}$, *we have*

$$u(a) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{a-x}{|a-x|^n} N(x) u(x) d\sigma(x) + \frac{1}{\omega_n} \iint_{\Omega} \frac{x-a}{|x-a|^n} (Du)(x) dx .$$

This extends the results in [Te], [Mo], [Bo], [BDS], [HL]. Moreover, a similar result is valid for the Martinelli-Bochner integral representation formula (cf. [HL]).

THEOREM 7.5. *Assume that* Ω *is an open subset of* $\mathbf{R}^n \subset \mathscr{A}_n$. *Let* u *be a locally integrable,* $\mathscr{A}_n$*-valued function which is also locally* $(n-1)$*-integrable on* Ω. *Let* $(C_\nu)_\nu$ *be an at most countable collection of closed subsets of* Ω *such that each pair* (u, C_ν) *satisfies one of the conditions* (α)-(γ) *stated in* §6. *Set* $A := \cup_\nu C_\nu$ *and also let* f *be a locally integrable* $\mathscr{A}_n$*-valued function on* Ω.

Assume that at least one of the following conditions holds:

(1) A *is closed,* u *is integrally continuous on* $\Omega \setminus A$ *and* $Du = f$ *in the distribution sense on* $\Omega \setminus A$;

(2) u *is Clifford differentiable at each point of* $\Omega \setminus A$ *and* $u'(x) = f(x)$ *for any* $x \in \Omega \setminus A$.

Then $Du = f$ *in the distribution sense on* Ω.

Note that, for $f = 0$, Theorem 7.3 gives sufficient conditions for u to be monogenic. These are substantially weaker than the ones presented in the literature (cf. e.g. [BDS]).

In our final application we briefly explain how the above theorem extends to more general linear first order differential operators. In doing so, it is convenient to slightly alter the definition of uniform locally $(n-1)$-integrability, and replace (4.1) by

$$\int_C |u| \, d\sigma < \varepsilon .$$

With this modification, the uniform locally $(n-1)$-integrability condition becomes invariant under multiplication with locally bounded functions.

Also, a locally $(n-1)$-integrable function will be called *locally integrally bounded* in Ω, if for any $K \in \text{comp}(\Omega)$ there exist $\theta, \kappa > 0$ such that for any Lipschitz $(n-1)$-dimensional submanifold C of $\mathbf{R}^n$, $C \subseteq K$, with $\mu_{n-1}(C) < \theta$ we have

$$\int_C |u| \, d\sigma < \kappa .$$

Consider now a linear, first order, differential operator

$$P = a_0(x) + \sum_{j=1}^{n} a_j(x) \partial_j ,$$

where the $\mathscr{A}_n$-valued functions $a_1, \ldots, a_n$ are locally Lipschitz continuous on Ω, and a_0 is a locally essentially bounded function on Ω. Let P^* stand for the formal transpose of P, i.e.

$$P^* = \left(a_0(x) - \sum_{j=1}^{n} (\partial_j a_j)(x) \right) + \sum_{j=1}^{n} a_j(x) \partial_j .$$

Also, for any $\xi \in \mathbf{R}^n$, the symbol of P is defined by $\sigma_P(\xi) := \sum_{j=1}^{n} \xi_j e_j a_j$. Recall that for two $\mathscr{A}_n$-valued, locally integrable functions u and f on Ω we have that $Pu = f$ in the distribution sense, if

$$\iint_\Omega P^*(\psi) u = \iint_\Omega \psi f$$

for any real-valued test function ψ on Ω.

Let u be a locally $(n-1)$-integrable function on Ω. We shall say that u is *P-differentiable at* $x \in \Omega$ provided that the limit

$$Pu(x) := \lim_{Q \downarrow x} \frac{1}{\lambda_n(Q)} \left\{ \iint_Q P^*(1) u + \int_{\partial Q} \sigma_P(N) u \, d\sigma \right\}$$

exists in $\mathscr{A}_n$. Proceeding as in Theorem 6.2, one can readily see that if u is actually differentiable at $x \in \Omega$, and if

$$\lim_{Q \downarrow x} \frac{1}{\lambda_n(Q)} \iint_Q | a_0(y) - a_0(x) | \, dy = 0 ,$$

then u is P-differentiable at x and $Pu(x) = a_0(x) u(x) + \sum_{j=1}^{n} a_j(x) \partial_j u(x)$.

The following result is an extension of Theorem 3.1.10 in [Hö].

THEOREM 7.6. *With the above definitions, consider* u, f *two locally integrable* $\mathscr{A}_n$*-valued functions on* Ω, *and let* $(C_\nu)_\nu$ *be an at most countable collection of closed subsets of* Ω. *Assume that* u *is also locally* *integrally bounded. Suppose that at least one of the following conditions holds:*

(1) for each ν, *the pair* (u, C_ν) *satisfies the condition* (α)*;*

(2) $a_0 \equiv 0$ *and for each* ν, *the pair* (u, C_ν) *satisfies one of the conditions* $(\alpha) - (\gamma)$.

Finally, set $A := \cup_\nu C_\nu$ *and assume that* u *is P-differentiable at each point of* $\Omega \setminus A$ *and that* $Pu(x) = f(x)$ *for any* $x \in \Omega \setminus A$. *Then* $Pu = f$ *in the distribution sense on* Ω.

Let us finally note that, due to the non-commutativity of the Clifford algebra $\mathscr{A}_n$ for $n \geqslant 3$, the results presented in this section are not in the most general form. For instance, one could consider the Clifford differentiation operator defined for *ordered pairs* of $\mathscr{A}_n$-valued functions (u, v) by

$$(u, v)' := \lim_{Q \downarrow a} \frac{1}{\lambda_n(Q)} \int_{\partial Q} u N v \, d\sigma \,,$$

for which all our techniques apply as well (cf. also [He1, 2]). However, we leave the details of this matter to the interested reader.

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