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as $\varphi(Q) = 0$ for any Q of the types (1)-(2) described above, and since φ is additive, it follows that $\psi(Q'_i) = \varphi(Q_i)$ for any $i \in J$. In particular,

$$\left| \sum_{i \in J} \psi(Q'_i) \right| = |\varphi(P)| \leq \left| \int_{\overset{\circ}{\Omega} \cap \partial P} u \right| + \left| \int_{\overset{\circ}{P} \cap b\Omega} u \right| + \left| \iint_{P \cap \Omega} f \right|.$$

By (4.2), the uniformly local $(n-1)$ -integrability of u , the integrability of u on $b\Omega$ and the integrability of f on Ω , the right hand side of the above equality can be made arbitrarily small, provided $\sum_{i \in J} \mu_{n-1}(Q'_i)$ is sufficiently small. However, since A' has Lebesgue measure zero in \mathbf{R}^{n-1} , this can be readily taken care of and this completes the proof of the theorem. \square

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally $(n-1)$ -integrable form u is *uniformly* $(n-1)$ -integrable only in a small neighborhood of $\mathcal{S}(u)$.

5. THE GLOBAL FORM OF THE STOKES FORMULA ON C^1 MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let M be a fixed, oriented, Hausdorff, differentiable manifold of class C^1 , and real dimension n .

DEFINITION 5.1. *A subset Ω of M is called a C^1 domain if for any $a \in \Omega \setminus \overset{\circ}{\Omega}$, there exist an open neighborhood U of a in M and a C^1 diffeomorphism $f = (f_1, f_2, \dots, f_n)$ of U onto an open neighborhood V of the origin in \mathbf{R}^n , such that*

$$U \cap \Omega = \{x \in U; f_n(x) \leq 0\}.$$

Clearly, the *border of the domain* Ω , $b\Omega := \Omega \setminus \overset{\circ}{\Omega}$ is either the empty set or a $(n-1)$ -dimensional C^1 -submanifold of M assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any C^1 domain is also a Lipschitz domain in \mathbf{R}^n .

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of \mathbf{R}^n . In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust

our previous definitions to the C^1 framework by carrying out the following simple modification. That is, whenever applicable, we shall replace “Lipschitz embedding” by “ C^1 -embedding”, i.e. Lipschitz embeddings which are C^1 functions. Note that, in particular, the condition (1.1) is in this case equivalent with the continuity of the functions

$$\frac{\partial h(s, x)}{\partial x_i} : S \times \omega \rightarrow \Omega, \quad i = 1, 2, \dots, n - 1.$$

Assuming this modification, all the previously introduced notions become invariant to C^1 diffeomorphisms and, hence, meaningful on C^1 manifolds. More specifically, we make the following.

DEFINITION 5.2. *Let Ω be a C^1 domain of M . A $(n - 1)$ -form u is said to be absolutely continuous (uniformly $(n - 1)$ -locally integrable) on Ω if for any point $P \in \Omega$ there exists a local coordinate map $h: U \rightarrow \mathbf{R}^n$ of M with $P \in U$ such that $(h^{-1})^*u$ is absolutely continuous (uniformly $(n - 1)$ -locally integrable, respectively) on $h(U \cap \Omega)$.*

Let u and f be locally integrable forms on M , having degrees $(n - 1)$ and n , respectively. Recall that $du = f$ on a open set Ω of M in the distribution sense, if for any $\varphi \in C_0^1(\Omega)$,

$$\int_M d\varphi \wedge u = - \int_M \varphi f.$$

THEOREM 5.3. *Let Ω be a C^1 domain of M , and u a $(n - 1)$ -form compactly supported in M . Assume that u is uniformly $(n - 1)$ -locally integrable and absolutely continuous on Ω , and that the singular set*

$$\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \text{supp } u$$

has $(n - 1)$ -dimensional Hausdorff measure zero.

If u is integrable on $\partial\Omega$ and du (taken in the sense of distribution theory) is integrable on Ω , then

$$\int_{\partial\Omega} u = \iint_{\Omega} du.$$

Proof. Using a smooth partition of unity and then working in local coordinates we can assume that $M = \mathbf{R}^n$. In this case, the conclusion is provided by Theorem 4.1. \square

Note that, here again it suffices to have the “uniform” part of the local $(n-1)$ -integrability condition for u fulfilled only on a small neighborhood of $\mathcal{S}(u)$ (cf. also Remark 4.3).

DEFINITION 5.4. *A closed subset A of M is said to have an almost regular boundary if A coincides with the closure of its interior and if there exist a family $(S_i)_{i \in I}$ of C^1 submanifolds of M and a locally finite family $(C_i)_{i \in I}$ of compact subsets of M such that:*

- (1) $C_i \subset S_i$, for any $i \in I$, and $\mathring{C}_i \cap \mathring{C}_j = \emptyset$, for all $i \neq j$ (the interiors are taken in S_i and in S_j , respectively);
- (2) $C_i \cap C_j$ has $(n-1)$ -dimensional Hausdorff measure zero for all $i \neq j$;
- (3) $\partial A = \cup_{i \in I} C_i$.

Note that if A has an almost regular boundary, then

$$\Omega := \mathring{A} \cup \left(\bigcup_{i \in I} \mathring{C}_i \right)$$

(the interior of A is taken in M) is a C^1 domain with border $b\Omega = \cup_{i \in I} \mathring{C}_i$. If u is an integrally continuous $(n-1)$ -form on M , it follows that u is integrable on each oriented submanifold \mathring{C}_i (with the standard orientation induced by \mathring{A}). Since ∂C_i has zero measure in S_i , we can define

$$\int_{\partial A} u := \sum_{i \in I} \int_{\mathring{C}_i} u$$

whenever $A \cap \text{supp } u$ is compact. Hence, without further proof, we can state the following.

THEOREM 5.5. *Let A be a subset of M with an almost regular boundary. If u is a $(n-1)$ -form which is uniformly $(n-1)$ -locally integrable on M , absolutely continuous on M , and for which $A \cap \text{supp } u$ is compact, then u is integrable on ∂A , du is integrable on \mathring{A} and*

$$\int_{\partial A} u = \iint_{\mathring{A}} du.$$