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## 3. SOME CALCULATIONS

In this section we give some computations of  $\chi_1(X)$  and  $\tilde{\chi}_1(X)$  which make use of explicit cell decompositions of the universal cover,  $\tilde{X}$ , of  $X$ . The simplest non-trivial example is the circle,  $X = S^1$ , which is treated in (A). In (B) we consider aspherical 2-complexes,  $X$ , arising from groups with two generators and one defining relation. In (C),  $X$  is a 3-dimensional lens space with odd order fundamental group; in fact, the computation there is already implicit in [GN<sub>1</sub>, §5(B)]. In (D),  $X$  is the real projective plane.

## (A) FINITE GRAPHS

A finite connected 1-complex,  $X$ , is aspherical so by Propositions 1.3 and 2.4,  $\Gamma = \pi_1(\mathcal{E}(X), \text{id})$  is trivial unless  $X$  has the homotopy type of  $S^1$ . Take  $X$  to be  $S^1$  with one 0-cell,  $v$ , and one 1-cell,  $e$ . Then  $\tilde{X}$  is the real line with the usual CW structure. Orient  $v$  by +1 and  $e$  by  $u \mapsto e^{2\pi i u}$ . Let  $t \in T \cong \pi_1(S^1, v)$  be represented by the loop  $u \mapsto e^{-2\pi i u}$  (this generator of  $T$  has been chosen for compatibility with §6). Recall that we use the right action of  $T$ , so

$$\tilde{\delta} = \begin{bmatrix} 0 & t-1 \\ 0 & 0 \end{bmatrix}.$$

The matrix  $\tilde{D}^{[R_1]}$  corresponding to positive rotation,  $R_1: S^1 \times I \rightarrow S^1$ , through  $2\pi$  (the first “tumble” in the language of §6) is

$$\tilde{D}^{[R_1]} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

note that the Sign Convention of §1 is used here. Thus  $\tilde{X}_1(S^1)([R_1])$  is represented by  $(t-1) \otimes 1$  which is homologous to  $t \otimes 1$ , and  $\chi_1(S^1)([R_1]) = \{t\}$ . Now,  $[R_1]$  generates the infinite cyclic group  $\Gamma$ . Making the standard identifications of  $\Gamma$  and  $T$  with  $\mathbf{Z}$  (i.e. identifying  $[R_1]$  and  $t^{-1}$  with  $1 \in \mathbf{Z}$ ), we obtain:

*Example 3.1.*  $\chi_1(S^1): \mathbf{Z} \rightarrow \mathbf{Z}$  is multiplication by  $-1$ .

*Remark.* The circle belongs to the classes of spaces considered in §4 and §6, so the methods there also apply.

## (B) GROUPS WITH TWO GENERATORS AND ONE RELATION

Let  $X$  be a finite 2-complex with one vertex,  $v$ , and one 2-cell,  $e^2$ . We further assume that  $X$  is aspherical. By Lyndon’s theorem [Ly], this is the case if and only if the element of the free group defined by the

attaching map of the 2-cell is not a proper power. As in (A), the group  $\Gamma \cong Z(\pi_1(X, v))$  is trivial unless  $X$  has two 1-cells,  $e_1^1$  and  $e_2^1$  (otherwise  $\chi(X) \neq 0$ ), so we assume this.

The case when  $X$  is homotopy equivalent to the 2-torus is exceptional. The following calculation is a special case of Example 6.15. Alternatively, the same result can be obtained by the method of Example 3.8 below. See also Corollary 4.8.

*Example 3.2.* Let  $X$  be homotopy equivalent to the 2-torus. Then  $\tilde{\chi}_1(X) = 0$ . Consequently, Proposition 2.8 implies  $\chi_1(X) = 0$ .

In all (aspherical) cases other than the 2-torus,  $\Gamma$  is known to be either trivial or infinite cyclic [Mu].

Orient  $v$  by  $+1$ , and choose orientations for the other cells. There is a corresponding presentation  $\langle x_1, x_2 \mid r \rangle$  of  $G = \pi_1(X, v)$ , where  $x_i$  denotes the element of  $G$  represented by the oriented loop  $e_i^1$ , and  $r$  is the attaching word in  $\{x_i^\pm\}$  with respect to the chosen orientation on  $e^2$ . Choose lifts of the cells so that:

$$\tilde{\partial}_1(\tilde{e}_i^1) = (x_i - 1)\tilde{v} \quad \text{and} \quad \tilde{\partial}_2(\tilde{e}^2) = \frac{\partial r}{\partial x_1} \tilde{e}_1^1 + \frac{\partial r}{\partial x_2} \tilde{e}_2^1.$$

We have written these in terms of the left action of  $G$  because we are using the free differential calculus [B, p. 45] which is traditionally done in terms of left actions. We will then convert to right actions using the involution  $*: \mathbf{Z}G \rightarrow \mathbf{Z}G$ ,  $\sum_i n_i g_i \mapsto \sum_i n_i g_i^{-1}$ .

For  $\gamma \in Z(G)$ , there is a unique (up to homotopy) cellular homotopy  $F^\gamma: \text{id}_X \rightarrow \text{id}_X$ . The track of the basepoint presents  $\gamma$  as a word in  $\{x_i^\pm\}$ , and

$$\tilde{D}_0^\gamma(\tilde{v}) = -\frac{\partial \gamma}{\partial x_1} \tilde{e}_1^1 - \frac{\partial \gamma}{\partial x_2} \tilde{e}_2^1.$$

There are  $\sigma_1, \sigma_2 \in \mathbf{Z}G$  such that  $\tilde{D}_1^\gamma(\tilde{e}_i) = \sigma_i \tilde{e}^2$ . Thus the relevant matrices are:

$$\tilde{\partial}_1 = [x_1^{-1} - 1 \ x_2^{-1} - 1], \quad \tilde{\partial}_2 = \begin{bmatrix} \left(\frac{\partial r}{\partial x_1}\right)^* \\ \left(\frac{\partial r}{\partial x_2}\right)^* \end{bmatrix}, \quad \tilde{D}_0 = \begin{bmatrix} -\left(\frac{\partial \gamma}{\partial x_1}\right)^* \\ -\left(\frac{\partial \gamma}{\partial x_2}\right)^* \end{bmatrix}.$$

and  $\tilde{D}_1 = [\sigma_1^* \ \sigma_2^*]$ . So  $\tilde{X}_1(X)(\gamma)$  is represented by the chain:

$$(3.3) \quad \text{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma) = \sum_{i=1}^2 \left[ (x_i^{-1} - 1) \otimes \left(\frac{\partial \gamma}{\partial x_i}\right)^* + \left(\frac{\partial r}{\partial x_i}\right)^* \otimes \sigma_i^* \right].$$

By Proposition 2.1, this implies:

$$\chi_1(X)(\gamma) = \sum_{i=1}^2 \left[ -\varepsilon\left(\frac{\partial\gamma}{\partial x_i}\right) A(x_i) - \varepsilon(\sigma_i) A\left(\frac{\partial r}{\partial x_i}\right) \right]$$

where  $\varepsilon: \mathbf{Z}G \rightarrow \mathbf{Z}$  is augmentation. For any  $g \in G$  represented by the word  $w$  in  $\{x_i^\pm\}$ ,  $A(g) = \sum_{j=1}^2 \varepsilon\left(\frac{\partial w}{\partial x_j}\right) A(x_j)$ . Substituting, we get:

$$\chi_1(X)(\gamma) = -A(\gamma) - \sum_{1 \leq i, j \leq 2} \varepsilon(\sigma_i) \varepsilon\left(\frac{\partial^2 r}{\partial x_j \partial x_i}\right) A(x_j).$$

The fact that  $\tilde{D}\tilde{\partial} - \tilde{\partial}\tilde{D} = \tilde{I}(1 - \eta_\#(\gamma)^{-1})$  yields six equations in  $\mathbf{Z}G$ . It is straightforward to check that when  $\varepsilon$  is applied to these they reduce to:

LEMMA 3.4. *For all  $1 \leq i, j \leq 2$ ,  $\varepsilon(\sigma_i) \varepsilon\left(\frac{\partial r}{\partial x_j}\right) = 0$ .*  $\square$

The chain complex  $C_*(X)$  is  $\mathbf{Z} \xrightarrow{\partial_2} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\partial_1} \mathbf{Z}$  where

$$\partial_2(1) = \left[ \varepsilon\left(\frac{\partial r}{\partial x_1}\right), \quad \varepsilon\left(\frac{\partial r}{\partial x_2}\right) \right]$$

and  $\partial_1 = 0$ . If  $H_2(X) = 0$  then  $\partial_2 \neq 0$ , and by Lemma 3.4,  $\varepsilon(\sigma_1) = \varepsilon(\sigma_2) = 0$ . Hence:

PROPOSITION 3.5. *If  $H_2(X) = 0$  then  $\chi_1(X) = -A$ .*  $\square$

If  $H_2(X) \neq 0$  then  $\partial_2 = 0$ . In this case we may regard  $A(x_1)$  and  $A(x_2)$  as a basis for the free abelian group  $G_{ab}$ . Writing  $H(r)$  for the *Fox Hessian* matrix of  $r$ , namely  $H(r)_{ij} = \varepsilon\left(\frac{\partial^2 r}{\partial x_i \partial x_j}\right)$ , and  $H(r)^t$  for its transpose we have:

PROPOSITION 3.6. *If  $H_2(X) \neq 0$  then*

$$\chi_1(X)(\gamma) = -A(\gamma) - [\varepsilon(\sigma_1) \ \varepsilon(\sigma_2)] H(r)^t \begin{bmatrix} A(x_1) \\ A(x_2) \end{bmatrix}. \quad \square$$

The matrix  $H(r)$  can be computed once we are given the relation  $r$ . The integers  $\varepsilon(\sigma_1)$  and  $\varepsilon(\sigma_2)$  depend on  $\gamma$ ; in general, they are hard to compute although we will do so in some special cases (see Examples 3.8 and 3.9 below).

The matrix  $H(r)$  is determined by the cup product  $H^1(X) \otimes H^1(X) \rightarrow H^2(X)$ :

**PROPOSITION 3.7.** *Assume  $H_2(X) \neq 0$ . Let  $\{\bar{A}(x_1), \bar{A}(x_2)\}$  be the dual basis for  $H^1(X)$ . Then  $H(r)_{ij} = (\bar{A}(x_i) \cup \bar{A}(x_j))([e^2])$ ; hence:  $\chi_1(X)(\gamma) = -A(\gamma) - (\bar{A}(x_1) \cup \bar{A}(x_2))([e^2])(\varepsilon(\sigma_1)A(x_2) - \varepsilon(\sigma_2)A(x_1))$ .*

*Proof.* This is the same formula given by Definition B<sub>1</sub> (note that  $H_*(X)$  is free abelian and so Definition B<sub>1</sub> applies to integral coefficients). A direct proof of Proposition 3.7 is also possible.  $\square$

*Example 3.8.*  $G = \langle x_1, x_2 \mid x_2 x_1^m x_2^{-1} x_1^{-m} \rangle$ ,  $m \geq 2$ . Here,  $Z(G)$  is generated by  $x_1^m$ , and  $H_2(X) \neq 0$ . One calculates:  $\frac{\partial r}{\partial x_1} = (x_2 - 1) \sum_{i=0}^{m-1} x_1^i$ ,

$$\frac{\partial r}{\partial x_2} = 1 - x_1^m, \quad \frac{\partial \gamma}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i, \quad \frac{\partial \gamma}{\partial x_2} = 0, \quad \sigma_1 = 0 \quad \text{and} \quad \sigma_2 = 1. \quad (\text{Actually,})$$

one sees these values for the sigmas intuitively and then one checks that the resulting  $\tilde{D}$  gives the right answer.) Thus  $\tilde{X}_1(X)(x_1^m)$  is represented by the cycle  $(x_1^{-1} - 1) \otimes \sum_{i=0}^{m-1} x_1^{-i} + (1 - x_1^{-m}) \otimes 1$  which is homologous to the canonical form:  $x_1^{-1} \otimes x_1 \left( \sum_{i=1}^{m-1} x_1^{-i} \right) + x_1^{m-1} \otimes x_1^{-(m-1)} x_1^{-m}$ . It follows that (see §2)  $\tilde{X}_1(X)(x_1^m) \in HH_1(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_1(Z(g_C))$  has  $[x_1^{-i}]$ -summand  $-\{x_1\} \in H_1(Z(x_1^{-i}))$ , for  $1 \leq i \leq m-1$ , and  $[x_1^{-m}]$ -summand  $(m-1)\{x_1\} \in H_1(G) = G_{ab}$ ; here,  $[g]$  denotes the conjugacy class of  $g$ . By Proposition 2.1 (or 3.6),  $\chi_1(X)(x_1^m) = 0$ . It is not difficult to see that  $\tilde{X}_1(X)$  is not an inner derivation. In particular, the first order Euler characteristic is zero, while  $\tilde{\chi}_1(X) \neq 0$ .

**EXAMPLE 3.9.**  $G = \langle x_1, x_2 \mid x_1^m x_2^n \rangle$ ,  $m \neq 0$  and  $n \neq 0$ . (If  $m$  and  $n$  are relatively prime, then  $G$  is the group of the  $(m, -n)$  torus knot.) Here,  $Z(G)$  is generated by  $x_1^m = x_2^{-n}$ , and  $H_2(X) = 0$ . By Proposition 3.5,  $\chi_1(X)(x_1^m) = -mA(x_1) = nA(x_2)$ . It is also of interest to calculate  $\tilde{X}_1(X)(x_1^m)$ .

We get  $\frac{\partial r}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$ ,  $\frac{\partial r}{\partial x_2} = x_1^m \sum_{i=0}^{n-1} x_2^i$ ,  $\frac{\partial \gamma}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$ ,  $\frac{\partial \gamma}{\partial x_2} = 0$ ,  $\sigma_1 = 0$

and  $\sigma_2 = x_2 - 1$ . Thus  $\tilde{X}_1(X)(x_1^m)$  is represented by the cycle  $(x_1^{-1} - 1) \otimes \sum_{i=0}^{m-1} x_1^{-i} + (\sum_{i=0}^{n-1} x_2^{-i}) x_1^{-m} \otimes (x_2^{-1} - 1)$  which is homologous to the canonical form:

$$\begin{aligned} \sum_{i=1}^{m-1} (x_1^{-1} \otimes x_1 x_1^{-i}) + \sum_{i=1}^{n-1} (x_2 \otimes x_2^{-1} x_2^i) + x_1^{m-1} \otimes x_1^{-(m-1)} x_1^{-m} \\ + x_2 \otimes x_2^{-1} 1. \end{aligned}$$

## (C) LENS SPACES

Let  $(p, q)$  be a pair of relatively prime positive integers with  $p > 1$ . The lens space  $L(p, q)$  is the orbit space of the action of the cyclic group  $\mathbf{Z}/p = \langle x \mid x^p = 1 \rangle$  on the 3-sphere  $S^3 = \{(z_0, z_1) \in \mathbf{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$  defined by  $x(z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi i q/p} z_1)$ . The point in  $L(p, q)$  determined by the orbit of  $(z_0, z_1) \in S^3$  will be denoted  $[z_0, z_1]$ .

For any pair of integers  $(m, n)$  such that  $m = n \bmod p$  define a smooth  $S^1$  action  $\gamma_{m, n}: S^1 \times L(p, q) \rightarrow L(p, q)$  by  $e^{2\pi i \theta} [z_0, z_1] = [e^{2\pi i \theta m/p} z_0, e^{2\pi i \theta nq/p} z_1]$ . These actions represent elements of  $\Gamma = \pi_1(\mathcal{E}(L(p, q)), \text{id})$ .

The group  $HH_1(\mathbf{Z}[\mathbf{Z}/p])$  is isomorphic to a direct sum of  $p$  copies of  $\mathbf{Z}/p$ ; furthermore, the Hochschild 1-cycles  $\{x \otimes x^{-1-k} \mid k = 0, \dots, p-1\}$  project to a set of generators for  $HH_1(\mathbf{Z}[\mathbf{Z}/p])$ . Define  $c_i, d_i \in \mathbf{Z}$  for  $0 \leq i \leq p-1$  by  $m - i - 1 = (c_i - 1)p + b_i$  and  $nq - i - 1 = (d_i - 1)p + b'_i$  where  $0 \leq b_i, b'_i \leq p-1$ . Let  $s_k = c_{k-1} + rd_{kq-1}$ , where the indices are interpreted mod  $p$  and  $rq = 1 \bmod p$ .

There is a natural cell structure on the universal cover,  $S^3$ , of  $L(p, q)$  (see [GN<sub>1</sub>, §5(B)]). Using this cell structure, [GN<sub>1</sub>, Lemma 5.3] asserts:

**PROPOSITION 3.10.**  $\tilde{\chi}_1(L(p, q))([\gamma_{m, n}]) \in HH_1(\mathbf{Z}[\mathbf{Z}/p])$  is represented by the Hochschild cycle  $- \sum_{k=0}^{p-1} s_k x \otimes x^{-1-k}$ .  $\square$

*Remark.* We take this opportunity to correct some inadvertently omitted minus signs from the computed examples in [GN<sub>1</sub>, §5]. In order to conform with our Sign Convention (see §1) used both here and in [GN<sub>1</sub>], the various chain homotopies  $\tilde{D}$  appearing in the explicit computations of [GN<sub>1</sub>, §5] should be replaced by  $-\tilde{D}$ . Consequently, in [GN<sub>1</sub>, Lemma 5.3], [GN<sub>1</sub>, Proposition 5.4] and [GN<sub>1</sub>, Corollary 5.5]  $\beta(\gamma_{m, n})$ ,  $R(\gamma_{m, n})$  and  $L(\gamma_{m, n})$  should be replaced by  $-\beta(\gamma_{m, n})$ ,  $-R(\gamma_{m, n})$  and  $-L(\gamma_{m, n})$  respectively. Similarly,  $R(F_n)$  should be replaced by  $-R(F_n)$  in [GN<sub>1</sub>, Theorem 5.1] and  $R(\Phi_2)$  should be replaced by  $-R(\Phi_2)$  in [GN<sub>1</sub>, §5(C)].

The homomorphism  $\varepsilon: HH_1(\mathbf{Z}[\mathbf{Z}/p]) \rightarrow H_1(\mathbf{Z}/p)$  takes the generators  $\{x \otimes x^{-1-k}\}$  to the same generator,  $\alpha$ , of  $H_1(\mathbf{Z}/p)$ . From the proof of [GN<sub>1</sub>, Corollary 5.5], we deduce:

**PROPOSITION 3.11.**  $\chi_1(L(p, q))([\gamma_{m, n}]) = -(m + n)\alpha$ .  $\square$

If  $p$  is odd then Propositions 3.10 and 3.11 give complete computations of  $\tilde{\chi}_1(L(p, q))$  and  $\chi_1(L(p, q))$  respectively because the  $[\gamma_{m, n}]$ 's generate  $\Gamma$ ;

indeed by [GN<sub>1</sub>, Proposition 5.7], for odd  $p$ ,  $\Gamma$  is cyclic of order  $2p^2$ . The proof there also shows that  $2[\gamma_{1,1}]$  is of order  $p^2$  and that  $p[\gamma_{0,p}]$  is of order 2 in  $\Gamma$ , so  $[\gamma_{2,2+p^2}]$  generates  $\Gamma$ .

#### (D) THE PROJECTIVE PLANE

We saw that when  $X$  is aspherical and  $\chi(X) \neq 0$  then  $\Gamma = 0$  and so our first order invariants vanish. In the presence of non-trivial higher homotopy these invariants need not vanish, despite  $\chi(X) \neq 0$ , as demonstrated by the example of the real projective plane  $X = P^2$ .

Write  $G \equiv \pi_1(P^2) \cong \mathbf{Z}/2$ ; denote the generator of  $G$  by  $t$ . Give  $P^2$  the customary cell structure consisting of one cell in each of dimensions 0, 1, and 2. The universal cover  $\tilde{P}^2$  is naturally identified with  $S^2$  and the corresponding cellular chain complex is:

$$C_2(S^2) \xrightarrow{1+t^{-1}} C_1(S^2) \xrightarrow{t^{-1}-1} C_0(S^2).$$

Every element of  $\Gamma$  can be represented by a basepoint preserving homotopy  $F: P^2 \times I \rightarrow P^2$  with  $F_0 = F_1 = \text{id}_{P^2}$ . We have  $\tilde{F}_0 = \tilde{F}_1 = \text{id}_{S^2}$  because the basepoint is preserved. It is easy to verify that the corresponding chain homotopy  $\tilde{D}_*: C_*(S^2) \rightarrow C_*(S^2)$  is then zero on  $C_0(S^2)$  and takes  $\tilde{e}_1$  to  $\tilde{e}_2 m(1 - t^{-1})$  where  $m \in \mathbf{Z}$ . By elementary obstruction theory, there exists  $F \equiv F^{(m)}$  realizing any  $m \in \mathbf{Z}$ . In this case  $\text{trace}(\tilde{\partial} \otimes \tilde{D}) = (1 + t^{-1}) \otimes m(1 - t^{-1})$  which is homologous to the canonical form  $mt^{-1} \otimes tt^{-1} - mt^{-1} \otimes tt^{-2}$ . Since  $\chi(P^2) = 1 \neq 0$ , the Gottlieb group  $\eta_*(\Gamma) \cong \mathcal{G}(P^2) = 0$  and so the derivation  $\tilde{\chi}_1(P^2)$  is a homomorphism and need not be distinguished from its cohomology class  $\tilde{\chi}_1(P^2) \in H^1(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2))) \cong \text{Hom}(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2)))$ . It follows that

$$\tilde{\chi}_1(P^2)([F^{(m)}]) = (m, -m) \in \mathbf{Z}/2 \oplus \mathbf{Z}/2 \cong HH_1(\mathbf{Z}(\mathbf{Z}/2)).$$

In particular, when  $m$  is odd  $\tilde{\chi}_1(P^2)([F^{(m)}]) \neq 0$ . On the other hand, this shows  $\chi_1(P^2) = 0$ .

## 4. $S^1$ -FIBRATIONS

In this section we investigate the first order Euler characteristic of the total space of an orientable Serre fibration with  $S^1$ -fiber.

Let  $S^1 \rightarrow X \xrightarrow{\pi} B$  be an orientable Serre fibration where  $B$  is a (not necessarily finite) connected CW complex and  $X$  has the homotopy type of a finite complex. By classical obstruction theory, fiber homotopy