3. Some calculations

In this section we give some computations of $\chi_1(X)$ and $\tilde{\chi}_1(X)$ which make use of explicit cell decompositions of the universal cover, $\tilde{X}$, of X. The simplest non-trivial example is the circle, $X = S^1$, which is treated in (A). In (B) we consider aspherical 2-complexes, X, arising from groups with two generators and one defining relation. In (C), X is a 3-dimensional lens space with odd order fundamental group; in fact, the computation there is already implicit in [GN_1, §5(B)]. In (D), X is the real projective plane.

(A) Finite graphs

A finite connected 1-complex, X, is aspherical so by Propositions 1.3 and 2.4, $\Gamma = \pi_1(\mathscr{E}(X), \mathrm{id})$ is trivial unless X has the homotopy type of S^1. Take X to be S^1 with one 0-cell, v, and one 1-cell, e. Then $\tilde{X}$ is the real line with the usual CW structure. Orient v by $+1$ and e by $u \mapsto e^{2\pi i u}$. Let $t \in T \equiv \pi_1(S^1, v)$ be represented by the loop $u \mapsto e^{-2\pi i u}$ (this generator of T has been chosen for compatibility with §6). Recall that we use the right action of T, so

$$\tilde{\partial} = \begin{bmatrix} 0 & t-1 \\ 0 & 0 \end{bmatrix}.$$

The matrix $\tilde{D}^{[R_1]}$ corresponding to positive rotation, $R_1 : S^1 \times I \to S^1$, through 2π (the first "tumble" in the language of §6) is

$$\tilde{D}^{[R_1]} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

note that the Sign Convention of §1 is used here. Thus $\tilde{\mathrm{X}}_1(S^1)([R_1])$ is represented by $(t-1) \otimes 1$ which is homologous to $t \otimes 1$, and $\chi_1(S^1)([R_1]) = \{t\}$. Now, $[R_1]$ generates the infinite cyclic group Γ. Making the standard identifications of Γ and T with $\mathbf{Z}$ (i.e. identifying $[R_1]$ and t^{-1} with $1 \in \mathbf{Z}$), we obtain:

Example 3.1. $\chi_1(S^1) : \mathbf{Z} \to \mathbf{Z}$ is multiplication by -1.

Remark. The circle belongs to the classes of spaces considered in §4 and §6, so the methods there also apply.

(B) Groups with two generators and one relation

Let X be a finite 2-complex with one vertex, v, and one 2-cell, e^2. We further assume that X is aspherical. By Lyndon's theorem [Ly], this is the case if and only if the element of the free group defined by the attaching map of the 2-cell is not a proper power. As in (A), the group $\Gamma \cong Z(\pi_1(X, v))$ is trivial unless X has two 1-cells, e_1^1 and e_2^1 (otherwise $\chi(X) \neq 0$), so we assume this.

The case when X is homotopy equivalent to the 2-torus is exceptional. The following calculation is a special case of Example 6.15. Alternatively, the same result can be obtained by the method of Example 3.8 below. See also Corollary 4.8.

Example 3.2. Let X be homotopy equivalent to the 2-torus. Then $\tilde{\chi}_1(X) = 0$. Consequently, Proposition 2.8 implies $\chi_1(X) = 0$.

In all (aspherical) cases other than the 2-torus, Γ is known to be either trivial or infinite cyclic [Mu].

Orient v by $+1$, and choose orientations for the the other cells. There is a corresponding presentation $\langle x_1, x_2 \mid r \rangle$ of $G = \pi_1(X, v)$, where x_i denotes the element of G represented by the oriented loop e_i^1, and r is the attaching word in $\{x_i^{\pm}\}$ with respect to the chosen orientation on e^2. Choose lifts of the cells so that:

$$\tilde{\partial}_1(\tilde{e}_i^1) = (x_i - 1)\tilde{v} \quad \text{and} \quad \tilde{\partial}_2(\tilde{e}^2) = \frac{\partial r}{\partial x_1}\tilde{e}_1^1 + \frac{\partial r}{\partial x_2}\tilde{e}_2^1 .$$

We have written these in terms of the left action of G because we are using the free differential calculus [B, p. 45] which is traditionally done in terms of left actions. We will then convert to right actions using the involution $*: \mathbf{Z}G \to \mathbf{Z}G$, $\sum_i n_i g_i \mapsto \sum_i n_i g_i^{-1}$.

For $\gamma \in Z(G)$, there is a unique (up to homotopy) cellular homotopy $F^\gamma: \mathrm{id}_X \to \mathrm{id}_X$. The track of the basepoint presents γ as a word in $\{x_i^{\pm}\}$, and

$$\tilde{D}_0^\gamma(\tilde{v}) = -\frac{\partial \gamma}{\partial x_1}\tilde{e}_1^1 - \frac{\partial \gamma}{\partial x_2}\tilde{e}_2^1 .$$

There are $\sigma_1, \sigma_2 \in \mathbf{Z}G$ such that $\tilde{D}_1^\gamma(\tilde{e}_i) = \sigma_i \tilde{e}^2$. Thus the relevant matrices are:

$$\tilde{\partial}_1 = [x_1^{-1} - 1 \;\; x_2^{-1} - 1], \quad \tilde{\partial}_2 = \begin{bmatrix} \left(\dfrac{\partial r}{\partial x_1}\right)^* \\ \left(\dfrac{\partial r}{\partial x_2}\right)^* \end{bmatrix}, \quad \tilde{D}_0 = \begin{bmatrix} -\left(\dfrac{\partial \gamma}{\partial x_1}\right)^* \\ -\left(\dfrac{\partial \gamma}{\partial x_2}\right)^* \end{bmatrix}.$$

and $\tilde{D}_1 = [\sigma_1^* \;\; \sigma_2^*]$. So $\mathrm{X}_1(X)(\gamma)$ is represented by the chain:

$$\text{(3.3)} \quad \operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma) = \sum_{i=1}^{2} \left[(x_i^{-1} - 1) \otimes \left(\frac{\partial \gamma}{\partial x_i}\right)^* + \left(\frac{\partial r}{\partial x_i}\right)^* \otimes \sigma_i^* \right].$$

By Proposition 2.1, this implies:

$$\chi_1(X)(\gamma) = \sum_{i=1}^{2} \left[-\varepsilon\left(\frac{\partial \gamma}{\partial x_i}\right) A(x_i) - \varepsilon(\sigma_i) A\left(\frac{\partial r}{\partial x_i}\right)\right]$$

where $\varepsilon: \mathbf{Z}G \to Z$ is augmentation. For any $g \in G$ represented by the word w in $\{x_i^{\pm}\}$, $A(g) = \sum_{j=1}^{2} \varepsilon\left(\frac{\partial w}{\partial x_j}\right) A(x_j)$. Substituting, we get:

$$\chi_1(X)(\gamma) = -A(\gamma) - \sum_{1 \leqslant i,j \leqslant 2} \varepsilon(\sigma_i)\varepsilon\left(\frac{\partial^2 r}{\partial x_j \partial x_i}\right) A(x_j)\,.$$

The fact that $\tilde{D}\tilde{\partial} - \tilde{\partial}\tilde{D} = \tilde{I}(1 - \eta_{\#}(\gamma)^{-1})$ yields six equations in $\mathbf{Z}G$. It is straightforward to check that when ε is applied to these they reduce to:

LEMMA 3.4. *For all* $1 \leqslant i, j \leqslant 2$, $\varepsilon(\sigma_i)\varepsilon\left(\frac{\partial r}{\partial x_j}\right) = 0$. □

The chain complex $C_*(X)$ is $\mathbf{Z} \xrightarrow{\partial_2} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\partial_1} \mathbf{Z}$ where

$$\partial_2(1) = \left[\varepsilon\left(\frac{\partial r}{\partial x_1}\right), \quad \varepsilon\left(\frac{\partial r}{\partial x_2}\right)\right]$$

and $\partial_1 = 0$. If $H_2(X) = 0$ then $\partial_2 \neq 0$, and by Lemma 3.4, $\varepsilon(\sigma_1) = \varepsilon(\sigma_2) = 0$. Hence:

PROPOSITION 3.5. *If* $H_2(X) = 0$ *then* $\chi_1(X) = -A$. □

If $H_2(X) \neq 0$ then $\partial_2 = 0$. In this case we may regard $A(x_1)$ and $A(x_2)$ as a basis for the free abelian group G_{ab}. Writing $H(r)$ for the *Fox Hessian* matrix of r, namely $H(r)_{ij} = \varepsilon\left(\frac{\partial^2 r}{\partial x_i \partial x_j}\right)$, and $H(r)^t$ for its transpose we have:

PROPOSITION 3.6. *If* $H_2(X) \neq 0$ *then*

$$\chi_1(X)(\gamma) = -A(\gamma) - [\varepsilon(\sigma_1)\ \varepsilon(\sigma_2)]H(r)^t \begin{bmatrix} A(x_1) \\ A(x_2) \end{bmatrix}. \quad \square$$

The matrix $H(r)$ can be computed once we are given the relation r. The integers $\varepsilon(\sigma_1)$ and $\varepsilon(\sigma_2)$ depend on γ; in general, they are hard to compute although we will do so in some special cases (see Examples 3.8 and 3.9 below).

The matrix $H(r)$ is determined by the cup product $H^1(X) \otimes H^1(X) \to H^2(X)$:

PROPOSITION 3.7. *Assume* $H_2(X) \neq 0$. *Let* $\{\bar{A}(x_1), \bar{A}(x_2)\}$ *be the dual basis for* $H^1(X)$. *Then* $H(r)_{ij} = (\bar{A}(x_i) \cup \bar{A}(x_j))([e^2])$; *hence:*

$$\chi_1(X)(\gamma) = -A(\gamma) - (\bar{A}(x_1) \cup \bar{A}(x_2))([e^2])(\varepsilon(\sigma_1)A(x_2) - \varepsilon(\sigma_2)A(x_1)) .$$

Proof. This is the same formula given by Definition B_1 (note that $H_*(X)$ is free abelian and so Definition B_1 applies to integral coefficients). A direct proof of Proposition 3.7 is also possible. □

Example 3.8. $G = \langle x_1, x_2 \,|\, x_2 x_1^m x_2^{-1} x_1^{-m} \rangle$, $m \geqslant 2$. Here, $Z(G)$ is generated by x_1^m, and $H_2(X) \neq 0$. One calculates: $\dfrac{\partial r}{\partial x_1} = (x_2 - 1) \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial r}{\partial x_2} = 1 - x_1^m$, $\dfrac{\partial \gamma}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial \gamma}{\partial x_2} = 0$, $\sigma_1 = 0$ and $\sigma_2 = 1$. (Actually, one sees these values for the sigmas intuitively and then one checks that the resulting $\tilde{D}$ gives the right answer.) Thus $\tilde{X}_1(X)(x_1^m)$ is represented by the cycle $(x_1^{-1} - 1) \otimes \sum_{i=0}^{m-1} x_1^{-i} + (1 - x_1^{-m}) \otimes 1$ which is homologous to the canonical form: $x_1^{-1} \otimes x_1 \quad (\sum_{i=1}^{m-1} x_1^{-i}) + x_1^{m-1} \otimes x_1^{-(m-1)} x_1^{-m}$. It follows that (see §2) $\tilde{X}_1(X)(x_1^m) \in HH_1(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_1(Z(g_C))$ has $[x_1^{-i}]$-summand $-\{x_1\} \in H_1(Z(x_1^{-i}))$, for $1 \leqslant i \leqslant m-1$, and $[x_1^{-m}]$-summand $(m-1)\{x_1\} \in H_1(G) = G_{\mathrm{ab}}$; here, $[g]$ denotes the conjugacy class of g. By Proposition 2.1 (or 3.6), $\chi_1(X)(x_1^m) = 0$. It is not difficult to see that $\tilde{X}_1(X)$ is not an inner derivation. In particular, the first order Euler characteristic is zero, while $\tilde{\chi}_1(X) \neq 0$.

EXAMPLE 3.9. $G = \langle x_1, x_2 \,|\, x_1^m x_2^n \rangle$, $m \neq 0$ and $n \neq 0$. (If m and n are relatively prime, then G is the group of the $(m, -n)$ torus knot.) Here, $Z(G)$ is generated by $x_1^m = x_2^{-n}$, and $H_2(X) = 0$. By Proposition 3.5, $\chi_1(X)(x_1^m) = -mA(x_1) = nA(x_2)$. It is also of interest to calculate $\tilde{X}_1(X)(x_1^m)$. We get $\dfrac{\partial r}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial r}{\partial x_2} = x_1^m \sum_{i=0}^{n-1} x_2^i$, $\dfrac{\partial \gamma}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial \gamma}{\partial x_2} = 0$, $\sigma_1 = 0$ and $\sigma_2 = x_2 - 1$. Thus $\tilde{X}_1(X)(x_1^m)$ is represented by the cycle $(x_1^{-1} - 1) \otimes \sum_{i=0}^{m-1} x_1^{-i} + (\sum_{i=0}^{n-1} x_2^{-i}) x_1^{-m} \otimes (x_2^{-1} - 1)$ which is homologous to the canonical form:

$$\sum_{i=1}^{m-1} (x_1^{-1} \otimes x_1 x_1^{-i}) + \sum_{i=1}^{n-1} (x_2 \otimes x_2^{-1} x_2^i) + x_1^{m-1} \otimes x_1^{-(m-1)} x_1^{-m}$$
$$+ x_2 \otimes x_2^{-1} 1 .$$

(C) LENS SPACES

Let (p, q) be a pair of relatively prime positive integers with $p > 1$. The lens space $L(p, q)$ is the orbit space of the action of the cyclic group $\mathbf{Z}/p = \langle x \mid x^p = 1 \rangle$ on the 3-sphere $S^3 = \{(z_0, z_1) \in \mathbf{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$ defined by $x(z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi i q/p} z_1)$. The point in $L(p, q)$ determined by the orbit of $(z_0, z_1) \in S^3$ will be denoted $[z_0, z_1]$.

For any pair of integers (m, n) such that $m = n \bmod p$ define a smooth S^1 action $\gamma_{m,n}: S^1 \times L(p, q) \to L(p, q)$ by $e^{2\pi i\theta}[z_0, z_1] = [e^{2\pi i\theta m/p} z_0, e^{2\pi i\theta nq/p} z_1]$. These actions represent elements of $\Gamma = \pi_1(\mathscr{E}(L(p, q)), \mathrm{id})$.

The group $HH_1(\mathbf{Z}[\mathbf{Z}/p])$ is isomorphic to a direct sum of p copies of $\mathbf{Z}/p$; furthermore, the Hochschild 1-cycles $\{x \otimes x^{-1-k} \mid k = 0, \ldots, p-1\}$ project to a set of generators for $HH_1(\mathbf{Z}[\mathbf{Z}/p])$. Define $c_i, d_i \in \mathbf{Z}$ for $0 \leqslant i \leqslant p-1$ by $m - i - 1 = (c_i - 1)p + b_i$ and $nq - i - 1 = (d_i - 1)p + b_i'$ where $0 \leqslant b_i, b_i' \leqslant p - 1$. Let $s_k = c_{k-1} + r d_{kq-1}$, where the indices are interpreted mod p and $rq = 1 \bmod p$.

There is a natural cell structure on the universal cover, S^3, of $L(p, q)$ (see [GN$_1$, §5(B)]). Using this cell structure, [GN$_1$, Lemma 5.3] asserts:

PROPOSITION 3.10. *$\tilde{\chi}_1(L(p, q))([\gamma_{m,n}]) \in HH_1(\mathbf{Z}[\mathbf{Z}/p])$ is represented by the Hochschild cycle* $-\sum_{k=0}^{p-1} s_k x \otimes x^{-1-k}$. □

Remark. We take this opportunity to correct some inadvertently omitted minus signs from the computed examples in [GN$_1$, §5]. In order to conform with our Sign Convention (see §1) used both here and in [GN$_1$], the various chain homotopies $\tilde{D}$ appearing in the explicit computations of [GN$_1$, §5] should be replaced by $-\tilde{D}$. Consequently, in [GN$_1$, Lemma 5.3], [GN$_1$, Proposition 5.4] and [GN$_1$, Corollary 5.5] $\beta(\gamma_{m,n})$, $R(\gamma_{m,n})$ and $L(\gamma_{m,n})$ should be replaced by $-\beta(\gamma_{m,n})$, $-R(\gamma_{m,n})$ and $-L(\gamma_{m,n})$ respectively. Similarly, $R(F_n)$ should be replaced by $-R(F_n)$ in [GN$_1$, Theorem 5.1] and $R(\Phi_2)$ should be replaced by $-R(\Phi_2)$ in [GN$_1$, §5(C)].

The homomorphism $\varepsilon: HH_1(\mathbf{Z}[\mathbf{Z}/p]) \to H_1(\mathbf{Z}/p)$ takes the generators $\{x \otimes x^{-1-k}\}$ to the same generator, α, of $H_1(\mathbf{Z}/p)$. From the proof of [GN$_1$, Corollary 5.5], we deduce:

PROPOSITION 3.11. $\chi_1(L(p, q))([\gamma_{m,n}]) = -(m+n)\alpha$. □

If p is odd then Propositions 3.10 and 3.11 give complete computations of $\tilde{\chi}_1(L(p, q))$ and $\chi_1(L(p, q))$ respectively because the $[\gamma_{m,n}]$'s generate Γ;

indeed by [GN$_1$, Proposition 5.7], for odd p, Γ is cyclic of order $2p^2$. The proof there also shows that $2[\gamma_{1,1}]$ is of order p^2 and that $p[\gamma_{0,p}]$ is of order 2 in Γ, so $[\gamma_{2,2+p^2}]$ generates Γ.

(D) THE PROJECTIVE PLANE

We saw that when X is aspherical and $\chi(X) \neq 0$ then $\Gamma = 0$ and so our first order invariants vanish. In the presence of non-trivial higher homotopy these invariants need not vanish, despite $\chi(X) \neq 0$, as demonstrated by the example of the real projective plane $X = P^2$.

Write $G \equiv \pi_1(P^2) \cong \mathbf{Z}/2$; denote the generator of G by t. Give P^2 the customary cell structure consisting of one cell in each of dimensions 0, 1, and 2. The universal cover $\tilde{P}^2$ is naturally identified with S^2 and the corresponding cellular chain complex is:

$$C_2(S^2) \xrightarrow{1+t^{-1}} C_1(S^2) \xrightarrow{t^{-1}-1} C_0(S^2)\,.$$

Every element of Γ can be represented by a basepoint preserving homotopy $F: P^2 \times I \to P^2$ with $F_0 = F_1 = \mathrm{id}_{P^2}$. We have $\tilde{F}_0 = \tilde{F}_1 = \mathrm{id}_{S^2}$ because the basepoint is preserved. It is easy to verify that the corresponding chain homotopy $\tilde{D}_*: C_*(S^2) \to C_*(S^2)$ is then zero on $C_0(S^2)$ and takes $\tilde{e}_1$ to $\tilde{e}_2 m(1-t^{-1})$ where $m \in \mathbf{Z}$. By elementary obstruction theory, there exists $F \equiv F^{(m)}$ realizing any $m \in \mathbf{Z}$. In this case $\mathrm{trace}(\tilde{\partial} \otimes \tilde{D}) = (1+t^{-1}) \otimes m(1-t^{-1})$ which is homologous to the canonical form $mt^{-1} \otimes tt^{-1} - mt^{-1} \otimes tt^{-2}$. Since $\chi(P^2) = 1 \neq 0$, the Gottlieb group $\eta_\#(\Gamma) \equiv \mathscr{G}(P^2) = 0$ and so the derivation $\tilde{\mathrm{X}}_1(P^2)$ is a homomorphism and need not be distinguished from its cohomology class $\tilde{\chi}_1(P^2) \in H^1(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2))) \cong \mathrm{Hom}(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2)))$. It follows that

$$\tilde{\chi}_1(P^2)\,([F^{(m)}]) = (m, -m) \in \mathbf{Z}/2 \oplus \mathbf{Z}/2 \cong HH_1(\mathbf{Z}(\mathbf{Z}/2))\,.$$

In particular, when m is odd $\tilde{\chi}_1(P^2)\,([F^{(m)}]) \neq 0$. On the other hand, this shows $\chi_1(P^2) = 0$.

4. S^1-FIBRATIONS

In this section we investigate the first order Euler characteristic of the total space of an orientable Serre fibration with S^1-fiber.

Let $S^1 \to X \xrightarrow{\pi} B$ be an orientable Serre fibration where B is a (not necessarily finite) connected CW complex and X has the homotopy type of a finite complex. By classical obstruction theory, fiber homotopy