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4. THE GLOBAL STOKES FORMULA FOR SIMPLE LIPSCHITZ DOMAINS IN  $\mathbf{R}^n$ 

A  $(n-1)$ -form  $u$  on  $\mathbf{R}^n$  is said to be *uniformly locally  $(n-1)$ -integrable on*  $\Omega \subseteq \mathbf{R}^n$  if it is locally  $(n-1)$ -integrable and, for any compact subset  $K$  of  $\mathbf{R}^n$  and any  $\varepsilon > 0$ , there exists a positive  $\delta = \delta(K, \varepsilon)$  such that

$$(4.1) \quad \left| \int_C u \right| < \varepsilon$$

whenever  $C$  is a  $(n-1)$ -dimensional Lipschitz submanifold  $C$  of  $\mathbf{R}^n$  which is contained in  $K \cap \Omega$  and has  $\mu_{n-1}(C) < \delta$ .

Examples include  $(n-1)$ -forms with locally bounded coefficients, or exhibiting isolated singularities of the type  $\|x\|^{-\alpha}$ ,  $\alpha < n-1$ .

Let us recall the notion of simple Lipschitz domain introduced in the last part of Definition 1.1. The main result of this section is the following.

**THEOREM 4.1.** *Let  $\Omega$  be a simple Lipschitz domain in  $\mathbf{R}^n$ , and let  $u$  be a compactly supported  $(n-1)$ -form in  $\mathbf{R}^n$  which is uniformly  $(n-1)$ -locally integrable on  $\mathbf{R}^n$ . Assume that  $u$  is absolutely continuous on  $\Omega$  and that the singular set*

$$\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \text{supp } u$$

*has  $(n-1)$ -dimensional Hausdorff measure zero.*

*Then, if  $u$  is integrable on  $b\Omega$  and  $du$  (in the distribution sense) is integrable on  $\Omega$ , we have*

$$\int_{b\Omega} u = \iint_{\mathring{\Omega}} du .$$

To prove this theorem, we shall need an auxiliary lemma. Two Lipschitz domains  $\Omega_1, \Omega_2$  in  $\mathbf{R}^n$  will be called *almost transversal* if  $\mu_{n-1}(b\Omega_1 \cap b\Omega_2) = 0$ . Let  $\Omega$  be a Lipschitz domain in  $\mathbf{R}^n$  and let  $\mathcal{R}$  stand for the collection of all rectangles of  $\mathbf{R}^n$  which are almost transversal to  $\Omega$ . Next, assume that  $u$  is a  $(n-1)$ -form compactly supported on  $\mathbf{R}^n$ , uniformly locally  $(n-1)$ -integrable on  $\mathbf{R}^n$ , and integrable on  $b\Omega$ . Also, let  $f$  be a locally integrable  $n$ -form on  $\mathbf{R}^n$  and consider the complex-valued mapping  $\varphi$  defined on  $\mathcal{R}$  by

$$\varphi(Q) := \int_{\mathring{\Omega} \cap b\Omega} U + \int_{\mathring{\Omega} \cap \partial\Omega} u - \iint_{Q \cap \Omega} f .$$

LEMMA 4.2. *Let  $\Omega, \mathcal{R}, u, f, \varphi$  be as above and assume that  $\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \text{supp } u$  has Hausdorff  $(n-1)$ -dimensional measure zero. Then the following hold.*

(1)  *$\mathcal{R}$  together with the usual subdivisions is a full rectangular system on  $\mathbf{R}^n$ .*

(2) *If  $P$  is a  $\mathcal{R}$ -paved set and  $(Q_i)_{i \in I}$  is a subdivision of  $P$ , then*

$$\sum_{i \in I} \varphi(Q_i) = \int_{\overset{\circ}{P} \cap b\Omega} u + \int_{\overset{\circ}{\Omega} \cap \partial P} u - \iint_{P \cap \Omega} f.$$

*In particular,  $\varphi$  is additive.*

(3) *The set  $\mathcal{S}(u)$  is  $(\varphi, 0)$ -negligible.*

*Proof.* For each  $k = 1, 2, \dots, n$ , let  $A_k$  be the collection of all  $c \in \mathbf{R}$  having the property that

$$\mu_{n-1}(\{x = (x_1, \dots, x_n) \in b\Omega; x_k = c\}) > 0.$$

Since  $\lambda_n(b\Omega) = 0$ , it follows by Fubini's theorem that  $A_k$  has Lebesgue measure zero in  $\mathbf{R}$  for any  $k$ .

Consider now  $Q, R_1, \dots, R_m \in \mathcal{R}$  such that  $R_v \subseteq Q$  for all  $v$ . Let  $(a_1, \dots, a_n)$  be the origin of  $Q$ , and  $(b_1, \dots, b_n)$  the end-point of  $Q$ . Similarly, for each  $v$ ,  $(a_1^v, \dots, a_n^v)$  will stand for the origin of  $R_v$ , whereas  $(b_1^v, \dots, b_n^v)$  will denote the end-point of  $R_v$ . The almost transversality hypothesis implies that  $a_k, b_k, a_k^v, b_k^v \in \mathbf{R} \setminus A_k$  for all  $v, k$ .

Now, since  $\lambda_1(A_k) = 0$ , for any a priory given  $\varepsilon > 0$ , we can select a finite sequence of real numbers  $x_{k, a_k}^v \in \mathbf{R} \setminus A_k$ ,  $a_k = 0, \dots, p_k$ , such that

$$\begin{aligned} a_k &= x_{k, 0}^v < \dots < x_{k, p_k}^v = b_k, \\ |x_{k, a_{k-1}}^v - x_{k, a_k}^v| &\leq \varepsilon n^{-1/2}, \end{aligned}$$

and, finally, so that  $a_k^v$  and  $b_k^v$  are among the numbers  $\{x_{k, a_k}^v\}_{a_k}$ . It is then easy to see that, for  $\varepsilon$  sufficiently small, the rectangles

$$Q_{(a_1, \dots, a_n)} := \prod_{k=1}^n [x_{k, a_{k-1}}^v, x_{k, a_k}^v], \quad \text{with } 1 \leq a_k \leq p_k,$$

form an elementary subdivision of  $Q$  which contains a subdivision of  $R_v$  for each  $1 \leq v \leq m$ . This completes the proof of (1).

Going further, (2) is immediate in the case in which the family  $(Q_i)_{i \in I}$  comes from an elementary subdivision of a larger rectangle containing  $P$ . Thus, the general case then easily follows from this and (1).

Next we turn our attention to (3). Fix  $Q \in \mathcal{R}$ ,  $K$  a compact subset of  $\Omega \setminus \mathcal{S}(u)$  and  $\varepsilon > 0$ . Since  $\mathcal{S}(u)$  has  $(n-1)$ -dimensional Hausdorff measure zero, it is thus possible to select finitely many rectangles  $R_1, \dots, R_m \in \mathcal{R}$  which do not intersect  $K$ , their interiors cover  $Q \cap \mathcal{S}(u)$ , and such that

$$\sum_{v=1}^m \mu_{n-1}(\partial R_v) < \varepsilon.$$

Then  $P := \cup_v (Q \cap R_v)$  is a  $\mathcal{R}$ -paved set contained in  $Q$  which does not intersect  $K$  and has the property that  $\mu_{n-1}(\partial P) < \varepsilon$ . Since  $\mathcal{R}$  is full, we can find an elementary subdivision  $(Q_i)_{i \in I}$  of  $Q$  and a subset  $J$  of  $I$  for which  $P = \cup_{i \in J} Q_i$ . In particular, we note that this implies  $Q_i \cap \mathcal{S}(u) = \emptyset$  for  $i \in I \setminus J$ . Using (2), we can write

$$\sum_{i \in J} \phi(Q_i) = \int_{\overset{\circ}{P} \cap b\Omega} u + \int_{\overset{\circ}{\Omega} \cap \partial P} u - \iint_P f.$$

Now, since  $u$  is integrable on  $b\Omega$  and  $f$  is integrable on  $\Omega$ , the first and the third terms from above can be made arbitrarily small by choosing  $K$  large enough. Furthermore, by taking  $\varepsilon$  sufficiently small and using the fact that  $u$  is uniformly locally  $(n-1)$ -integrable, the second term can also be made arbitrarily small. The proof of the lemma is therefore finished.  $\square$

*Proof of Theorem 4.1.* Since in the conclusion of the theorem  $u$  intervenes only through its values on  $\Omega$ , there is no loss of generality assuming that  $u = 0$  on  $\mathbf{R}^n \setminus \bar{\Omega}$ , i.e. that  $\text{supp } u \subseteq \bar{\Omega}$  (note that this does not alter the hypotheses either). We set  $f := du$  in  $\overset{\circ}{\Omega}$ , zero in  $\mathbf{R}^n \setminus \overset{\circ}{\Omega}$ , and adopt the notation introduced in Lemma 4.2. Clearly, it is enough to prove that  $\phi(Q) = 0$  for any  $Q \in \mathcal{R}$ . First, let us observe that from (the proof of) Theorem 1.3 this is immediate for rectangles of the following two types:

- (1)  $Q \subset \overset{\circ}{\Omega}$  or  $u = 0$  on  $Q$ ;
- (2) after suitably permuting the coordinates in  $\mathbf{R}^n$ ,

$$Q \cap \Omega = \{x = (x', x_n); x' \in Q' \text{ and } a_n \leq x_n \leq \theta(x') < b_n\},$$

where  $Q = Q' \times [a_n, b_n]$  and  $\theta: \mathbf{R}^{n-1} \rightarrow (a_n, b_n)$  is a Lipschitz function. On the other hand, the compact set  $\mathcal{S}(u)$  has zero  $\mu_{n-1}$ -measure and, hence, by Lemma 4.2, is  $(\phi, 0)$ -negligible. Consequently, using Theorem 3.4 with  $s = t = 0$ , it suffices to show that any point  $a \in b\Omega$  has an open neighborhood  $\mathcal{U}$  in  $\mathbf{R}^n$  such that  $\phi(R) = 0$  for all rectangles  $R \in \mathcal{R}$  included in  $\mathcal{U}$  and containing  $a$ . By possibly relabeling the coordinates first, we can

find an open rectangle  $U$  in  $\mathbf{R}^n$  and a Lipschitz function  $\theta: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$  such that

$$U \cap \Omega = U \cap \{x = (x', x_n); x_n \leq \theta(x')\}.$$

Now let  $R = R' \times [a_n, b_n] \in \mathcal{R}$  be a fixed rectangle contained in  $U$ , where  $R'$  is a rectangle in  $\mathbf{R}^{n-1}$  and  $a_n, b_n \in \mathbf{R}$ ,  $a_n < b_n$ . Denote by  $\mathcal{R}'$  the collection of all rectangles  $Q'$  from  $\mathbf{R}^{n-1}$  which are contained in  $R'$ , having  $p(Q') \leq p(R') + 1$  and such that  $Q' \times [a_n, b_n] \in \mathcal{R}$ . Then, with the usual subdivisions,  $(\mathcal{R}', \text{div})$  becomes a rectangular system on  $R'$ .

Next, we introduce the mapping  $\psi: \mathcal{R}' \rightarrow \mathbf{C}$  by setting

$$\psi(Q') := \varphi(Q' \times [a_n, b_n]), \quad Q' \in \mathcal{R}'$$

Thus, everything comes down to proving that  $\psi$  vanishes identically on  $\mathcal{R}'$ . Let us consider the following compact set in  $\mathbf{R}^n$ :

$$A' := R' \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)).$$

If a rectangle  $Q' \in \mathcal{R}'$  does not meet  $A'$ , then the rectangle  $Q' \times [a_n, b_n] \in \mathcal{R}$  is either of type (1) or (2) from above, so that, at any rate,  $\psi(Q') = 0$ .

Since  $\varphi$  is additive, so is  $\psi$  and, by the equivalence  $(1) \Leftrightarrow (3)$  in Theorem 3.4 with  $s = t = 0$ , it suffices to prove that  $A'$  is  $(\psi, 0)$ -negligible. To this end, let  $Q' \in \mathcal{R}'$  and let  $(Q'_i)_{i \in I}$  be a subdivision of  $Q'$  such that  $\delta_i := \text{diam}(Q'_i) \leq \delta$ , for all  $i$ , for some positive  $\delta$ . We also introduce

$$J := \{i \in I; Q'_i \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)) \neq \emptyset\}.$$

For each  $i \in J$  we have that at least one of the sets  $Q'_i \cap \theta^{-1}(a_n)$ ,  $Q'_i \cap \theta^{-1}(b_n)$  is empty provided  $\delta$  is sufficiently small. Assuming that this is the case, we set

$$Q_i := Q'_i \times [a_n, a_n + \delta_i M]$$

if  $Q'_i \cap \theta^{-1}(a_n) \neq \emptyset$ , and

$$Q_i := Q'_i \times [b_n - \delta_i M, b_n],$$

if  $Q'_i \cap \theta^{-1}(b_n) \neq \emptyset$ . Here  $M$  stands for the (essential) supremum of  $|\nabla \theta|$  on  $R'$ . Then  $P := \cup_{i \in J} Q_i$  is a  $\mathcal{R}$ -paved set having

$$(4.2) \quad \mu_{n-1}(\partial P) \leq C \sum_{i \in J} \mu_{n-1}(Q'_i)$$

for some positive constant  $C$  depending exclusively on  $\theta$  and  $R'$ . Furthermore,

as  $\varphi(Q) = 0$  for any  $Q$  of the types (1)-(2) described above, and since  $\varphi$  is additive, it follows that  $\psi(Q'_i) = \varphi(Q_i)$  for any  $i \in J$ . In particular,

$$\left| \sum_{i \in J} \psi(Q'_i) \right| = |\varphi(P)| \leq \left| \int_{\overset{\circ}{\Omega} \cap \partial P} u \right| + \left| \int_{\overset{\circ}{P} \cap b\Omega} u \right| + \left| \iint_{P \cap \Omega} f \right|.$$

By (4.2), the uniformly local  $(n-1)$ -integrability of  $u$ , the integrability of  $u$  on  $b\Omega$  and the integrability of  $f$  on  $\Omega$ , the right hand side of the above equality can be made arbitrarily small, provided  $\sum_{i \in J} \mu_{n-1}(Q'_i)$  is sufficiently small. However, since  $A'$  has Lebesgue measure zero in  $\mathbf{R}^{n-1}$ , this can be readily taken care of and this completes the proof of the theorem.  $\square$

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally  $(n-1)$ -integrable form  $u$  is *uniformly*  $(n-1)$ -integrable only in a small neighborhood of  $\mathcal{S}(u)$ .

## 5. THE GLOBAL FORM OF THE STOKES FORMULA ON $C^1$ MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let  $M$  be a fixed, oriented, Hausdorff, differentiable manifold of class  $C^1$ , and real dimension  $n$ .

DEFINITION 5.1. A subset  $\Omega$  of  $M$  is called a  $C^1$  domain if for any  $a \in \Omega \setminus \overset{\circ}{\Omega}$ , there exist an open neighborhood  $U$  of  $a$  in  $M$  and a  $C^1$  diffeomorphism  $f = (f_1, f_2, \dots, f_n)$  of  $U$  onto an open neighborhood  $V$  of the origin in  $\mathbf{R}^n$ , such that

$$U \cap \Omega = \{x \in U; f_n(x) \leq 0\}.$$

Clearly, the *border of the domain*  $\Omega$ ,  $b\Omega := \Omega \setminus \overset{\circ}{\Omega}$  is either the empty set or a  $(n-1)$ -dimensional  $C^1$ -submanifold of  $M$  assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any  $C^1$  domain is also a Lipschitz domain in  $\mathbf{R}^n$ .

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of  $\mathbf{R}^n$ . In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust