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Autor: Jurchescu, Martin / Mitrea, Marius
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4. THE GLOBAL STOKES FORMULA FOR SIMPLE LIPSCHITZ DOMAINS IN \mathbf{R}^n

A $(n-1)$ -form u on \mathbf{R}^n is said to be *uniformly locally $(n-1)$ -integrable* on $\Omega \subseteq \mathbf{R}^n$ if it is locally $(n-1)$ -integrable and, for any compact subset K of \mathbf{R}^n and any $\varepsilon > 0$, there exists a positive $\delta = \delta(K, \varepsilon)$ such that

$$(4.1) \quad \left| \int_C u \right| < \varepsilon$$

whenever C is a $(n-1)$ -dimensional Lipschitz submanifold C of \mathbf{R}^n which is contained in $K \cap \Omega$ and has $\mu_{n-1}(C) < \delta$.

Examples include $(n-1)$ -forms with locally bounded coefficients, or exhibiting isolated singularities of the type $\|x\|^{-\alpha}$, $\alpha < n-1$.

Let us recall the notion of simple Lipschitz domain introduced in the last part of Definition 1.1. The main result of this section is the following.

THEOREM 4.1. *Let Ω be a simple Lipschitz domain in \mathbf{R}^n , and let u be a compactly supported $(n-1)$ -form in \mathbf{R}^n which is uniformly $(n-1)$ -locally integrable on \mathbf{R}^n . Assume that u is absolutely continuous on Ω and that the singular set*

$$\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \text{supp } u$$

has $(n-1)$ -dimensional Hausdorff measure zero.

Then, if u is integrable on $b\Omega$ and du (in the distribution sense) is integrable on Ω , we have

$$\int_{b\Omega} u = \iint_{\overset{\circ}{\Omega}} du.$$

To prove this theorem, we shall need an auxiliary lemma. Two Lipschitz domains Ω_1, Ω_2 in \mathbf{R}^n will be called *almost transversal* if $\mu_{n-1}(b\Omega_1 \cap b\Omega_2) = 0$. Let Ω be a Lipschitz domain in \mathbf{R}^n and let \mathcal{R} stand for the collection of all rectangles of \mathbf{R}^n which are almost transversal to Ω . Next, assume that u is a $(n-1)$ -form compactly supported on \mathbf{R}^n , uniformly locally $(n-1)$ -integrable on \mathbf{R}^n , and integrable on $b\Omega$. Also, let f be a locally integrable n -form on \mathbf{R}^n and consider the complex-valued mapping φ defined on \mathcal{R} by

$$\varphi(Q) := \int_{\overset{\circ}{Q} \cap b\Omega} U + \int_{\overset{\circ}{Q} \cap \partial Q} u - \iint_{Q \cap \Omega} f.$$

LEMMA 4.2. Let $\Omega, \mathcal{R}, u, f, \varphi$ be as above and assume that $\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \text{supp } u$ has Hausdorff $(n-1)$ -dimensional measure zero. Then the following hold.

(1) \mathcal{R} together with the usual subdivisions is a full rectangular system on \mathbf{R}^n .

(2) If P is a \mathcal{R} -paved set and $(Q_i)_{i \in I}$ is a subdivision of P , then

$$\sum_{i \in I} \varphi(Q_i) = \int_{\overset{\circ}{P} \cap b\Omega} u + \int_{\overset{\circ}{\Omega} \cap \partial P} u - \iint_{P \cap \Omega} f.$$

In particular, φ is additive.

(3) The set $\mathcal{S}(u)$ is $(\varphi, 0)$ -negligible.

Proof. For each $k = 1, 2, \dots, n$, let A_k be the collection of all $c \in \mathbf{R}$ having the property that

$$\mu_{n-1}(\{x = (x_1, \dots, x_n) \in b\Omega; x_k = c\}) > 0.$$

Since $\lambda_n(b\Omega) = 0$, it follows by Fubini's theorem that A_k has Lebesgue measure zero in \mathbf{R} for any k .

Consider now $Q, R_1, \dots, R_m \in \mathcal{R}$ such that $R_v \subseteq Q$ for all v . Let (a_1, \dots, a_n) be the origin of Q , and (b_1, \dots, b_n) the end-point of Q . Similarly, for each v , (a_1^v, \dots, a_n^v) will stand for the origin of R_v , whereas (b_1^v, \dots, b_n^v) will denote the end-point of R_v . The almost transversality hypothesis implies that $a_k, b_k, a_k^v, b_k^v \in \mathbf{R} \setminus A_k$ for all v, k .

Now, since $\lambda_1(A_k) = 0$, for any a priori given $\varepsilon > 0$, we can select a finite sequence of real numbers $x_{k, \alpha_k}^v \in \mathbf{R} \setminus A_k$, $\alpha_k = 0, \dots, p_k$, such that

$$a_k = x_{k, 0}^v < \dots < x_{k, p_k}^v = b_k, \\ |x_{k, \alpha_{k-1}}^v - x_{k, \alpha_k}^v| \leq \varepsilon n^{-1/2},$$

and, finally, so that a_k^v and b_k^v are among the numbers $\{x_{k, \alpha_k}^v\}_{\alpha_k}$. It is then easy to see that, for ε sufficiently small, the rectangles

$$Q_{(\alpha_1, \dots, \alpha_n)} := \prod_{k=1}^n [x_{k, \alpha_{k-1}}, x_{k, \alpha_k}], \quad \text{with } 1 \leq \alpha_k \leq p_k,$$

form an elementary subdivision of Q which contains a subdivision of R_v for each $1 \leq v \leq m$. This completes the proof of (1).

Going further, (2) is immediate in the case in which the family $(Q_i)_{i \in I}$ comes from an elementary subdivision of a larger rectangle containing P . Thus, the general case then easily follows from this and (1).

Next we turn our attention to (3). Fix $Q \in \mathcal{R}$, K a compact subset of $\Omega \setminus \mathcal{S}(u)$ and $\varepsilon > 0$. Since $\mathcal{S}(u)$ has $(n-1)$ -dimensional Hausdorff measure zero, it is thus possible to select finitely many rectangles $R_1, \dots, R_m \in \mathcal{R}$ which do not intersect K , their interiors cover $Q \cap \mathcal{S}(u)$, and such that

$$\sum_{v=1}^m \mu_{n-1}(\partial R_v) < \varepsilon.$$

Then $P := \cup_v (Q \cap R_v)$ is a \mathcal{R} -paved set contained in Q which does not intersect K and has the property that $\mu_{n-1}(\partial P) < \varepsilon$. Since \mathcal{R} is full, we can find an elementary subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $P = \cup_{i \in J} Q_i$. In particular, we note that this implies $Q_i \cap \mathcal{S}(u) = \emptyset$ for $i \in I \setminus J$. Using (2), we can write

$$\sum_{i \in J} \varphi(Q_i) = \int_{\overset{\circ}{P} \cap b\Omega} u + \int_{\overset{\circ}{Q} \cap \partial P} u - \iint_P f.$$

Now, since u is integrable on $b\Omega$ and f is integrable on Ω , the first and the third terms from above can be made arbitrarily small by choosing K large enough. Furthermore, by taking ε sufficiently small and using the fact that u is uniformly locally $(n-1)$ -integrable, the second term can also be made arbitrarily small. The proof of the lemma is therefore finished. \square

Proof of Theorem 4.1. Since in the conclusion of the theorem u intervenes only through its values on Ω , there is no loss of generality assuming that $u = 0$ on $\mathbf{R}^n \setminus \bar{\Omega}$, i.e. that $\text{supp } u \subseteq \bar{\Omega}$ (note that this does not alter the hypotheses either). We set $f := du$ in $\overset{\circ}{\Omega}$, zero in $\mathbf{R}^n \setminus \overset{\circ}{\Omega}$, and adopt the notation introduced in Lemma 4.2. Clearly, it is enough to prove that $\varphi(Q) = 0$ for any $Q \in \mathcal{R}$. First, let us observe that from (the proof of) Theorem 1.3 this is immediate for rectangles of the following two types:

- (1) $Q \subset \overset{\circ}{\Omega}$ or $u = 0$ on Q ;
- (2) after suitably permuting the coordinates in \mathbf{R}^n ,

$$Q \cap \Omega = \{x = (x', x_n); x' \in Q' \text{ and } a_n \leq x_n \leq \theta(x') < b_n\},$$

where $Q = Q' \times [a_n, b_n]$ and $\theta: \mathbf{R}^{n-1} \rightarrow (a_n, b_n)$ is a Lipschitz function. On the other hand, the compact set $\mathcal{S}(u)$ has zero μ_{n-1} -measure and, hence, by Lemma 4.2, is $(\varphi, 0)$ -negligible. Consequently, using Theorem 3.4 with $s = t = 0$, it suffices to show that any point $a \in b\Omega$ has an open neighborhood \mathcal{U} in \mathbf{R}^n such that $\varphi(R) = 0$ for all rectangles $R \in \mathcal{R}$ included in \mathcal{U} and containing a . By possibly relabeling the coordinates first, we can

find an open rectangle U in \mathbf{R}^n and a Lipschitz function $\theta: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ such that

$$U \cap \Omega = U \cap \{x = (x', x_n); x_n \leq \theta(x')\}.$$

Now let $R = R' \times [a_n, b_n] \in \mathcal{R}$ be a fixed rectangle contained in U , where R' is a rectangle in \mathbf{R}^{n-1} and $a_n, b_n \in \mathbf{R}, a_n < b_n$. Denote by \mathcal{R}' the collection of all rectangles Q' from \mathbf{R}^{n-1} which are contained in R' , having $p(Q') \leq p(R') + 1$ and such that $Q' \times [a_n, b_n] \in \mathcal{R}$. Then, with the usual subdivisions, $(\mathcal{R}', \text{div})$ becomes a rectangular system on R' .

Next, we introduce the mapping $\psi: \mathcal{R}' \rightarrow \mathbf{C}$ by setting

$$\psi(Q') := \varphi(Q' \times [a_n, b_n]), \quad Q' \in \mathcal{R}'$$

Thus, everything comes down to proving that ψ vanishes identically on \mathcal{R}' . Let us consider the following compact set in \mathbf{R}^n :

$$A' := R' \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)).$$

If a rectangle $Q' \in \mathcal{R}'$ does not meet A' , then the rectangle $Q' \times [a_n, b_n] \in \mathcal{R}$ is either of type (1) or (2) from above, so that, at any rate, $\psi(Q') = 0$.

Since φ is additive, so is ψ and, by the equivalence (1) \Leftrightarrow (3) in Theorem 3.4 with $s = t = 0$, it suffices to prove that A' is $(\psi, 0)$ -negligible. To this end, let $Q' \in \mathcal{R}'$ and let $(Q'_i)_{i \in I}$ be a subdivision of Q' such that $\delta_i := \text{diam}(Q'_i) \leq \delta$, for all i , for some positive δ . We also introduce

$$J := \{i \in I; Q'_i \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)) \neq \emptyset\}.$$

For each $i \in J$ we have that at least one of the sets $Q'_i \cap \theta^{-1}(a_n)$, $Q'_i \cap \theta^{-1}(b_n)$ is empty provided δ is sufficiently small. Assuming that this is the case, we set

$$Q_i := Q'_i \times [a_n, a_n + \delta_i M]$$

if $Q'_i \cap \theta^{-1}(a_n) \neq \emptyset$, and

$$Q_i := Q'_i \times [b_n - \delta_i M, b_n],$$

if $Q'_i \cap \theta^{-1}(b_n) \neq \emptyset$. Here M stands for the (essential) supremum of $|\nabla \theta|$ on R' . Then $P := \cup_{i \in J} Q_i$ is a \mathcal{R} -paved set having

$$(4.2) \quad \mu_{n-1}(\partial P) \leq C \sum_{i \in J} \mu_{n-1}(Q'_i)$$

for some positive constant C depending exclusively on θ and R' . Furthermore,

as $\varphi(Q) = 0$ for any Q of the types (1)-(2) described above, and since φ is additive, it follows that $\psi(Q'_i) = \varphi(Q_i)$ for any $i \in J$. In particular,

$$\left| \sum_{i \in J} \psi(Q'_i) \right| = |\varphi(P)| \leq \left| \int_{\overset{\circ}{\Omega} \cap \partial P} u \right| + \left| \int_{\overset{\circ}{P} \cap b\Omega} u \right| + \left| \iint_{P \cap \Omega} f \right|.$$

By (4.2), the uniformly local $(n-1)$ -integrability of u , the integrability of u on $b\Omega$ and the integrability of f on Ω , the right hand side of the above equality can be made arbitrarily small, provided $\sum_{i \in J} \mu_{n-1}(Q'_i)$ is sufficiently small. However, since A' has Lebesgue measure zero in \mathbf{R}^{n-1} , this can be readily taken care of and this completes the proof of the theorem. \square

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally $(n-1)$ -integrable form u is *uniformly* $(n-1)$ -integrable only in a small neighborhood of $\mathcal{S}(u)$.

5. THE GLOBAL FORM OF THE STOKES FORMULA ON C^1 MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let M be a fixed, oriented, Hausdorff, differentiable manifold of class C^1 , and real dimension n .

DEFINITION 5.1. *A subset Ω of M is called a C^1 domain if for any $a \in \Omega \setminus \overset{\circ}{\Omega}$, there exist an open neighborhood U of a in M and a C^1 diffeomorphism $f = (f_1, f_2, \dots, f_n)$ of U onto an open neighborhood V of the origin in \mathbf{R}^n , such that*

$$U \cap \Omega = \{x \in U; f_n(x) \leq 0\}.$$

Clearly, the *border of the domain* Ω , $b\Omega := \Omega \setminus \overset{\circ}{\Omega}$ is either the empty set or a $(n-1)$ -dimensional C^1 -submanifold of M assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any C^1 domain is also a Lipschitz domain in \mathbf{R}^n .

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of \mathbf{R}^n . In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust