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Hence, on account of the continuity of ρ , we see that $\int_Q d\mu = \int_{\partial Q} u$, for any Q . With this at hand and by once again using the hypothesis (4), we conclude that μ is absolutely continuous with respect to the n -dimensional Lebesgue measure λ_n . Therefore, if $f \in L^1(\Omega, \text{loc})$ denotes the Radon-Nikodym-Lebesgue density of μ with respect to λ_n , we have that

$$\int_{\partial Q} u = \int_Q d\mu = \iint_Q f$$

for any $Q \in \mathcal{R}(\Omega)$. Using this, Theorem 1.3 finally implies that $du = f$ in the distribution sense on Ω , and this concludes the proof of the theorem. \square

REMARK 2.2. Inspection of the proof also shows that $\rho(K) = \int_K |f| d\lambda_n$ for any compact subset K of Ω , and that $|f| \leq g$ a.e. on Ω .

An *integrally Lipschitz* $(n-1)$ -form in Ω is a locally $(n-1)$ -integrable form u for which there exists $M > 0$ so that

$$\left| \int_{\partial Q} u \right| \leq M \lambda_n(Q)$$

for each $Q \in \mathcal{R}(\Omega)$. Note that any integrally Lipschitz $(n-1)$ -form u in Ω satisfies the equivalent conditions in Theorem 1.3.

3. A MAXIMUM PRINCIPLE

The purpose of this section is to prove a localization theorem which plays a significant role in the sequel. Here, our approach is of an abstract nature.

Let \mathcal{X} be a fixed metric space. In general, for an arbitrary set E , we shall denote by $\mathcal{F}(E)$ the collection of all finite families of subsets of E , and by $\mathcal{S}(E)$ the collection of all subsets of $\mathcal{F}(E)$.

DEFINITION 3.1. A *rectangular system* on \mathcal{X} is a subset \mathcal{R} of $\text{comp}(\mathcal{X})$ together with an application $\text{div}: \mathcal{R} \rightarrow \mathcal{S}(\mathcal{R})$ satisfying the following:

- (1) If $Q \in \mathcal{R}$ and $(Q_i)_{i \in I} \in \text{div}(Q)$, then $Q_i \subseteq Q$ for any $i \in I$;
- (2) For any $Q \in \mathcal{R}$ and any $\varepsilon > 0$, there exists $(Q_i)_{i \in I} \in \text{div}(Q)$ so that $\text{diam}(Q_i) < \varepsilon$ for every $i \in I$.

The elements of \mathcal{R} will be called *rectangles*, whereas the elements of $\text{div}(Q)$, for $Q \in \mathcal{R}$, will be called the *subdivisions* of Q . Later, we shall also need the following.

DEFINITION 3.2. A rectangular system $(\mathcal{R}, \text{div})$ is said to be full if for any $Q \in \mathcal{R}$ and any $R_1, \dots, R_m \in \mathcal{R}$ with $R_v \subseteq Q, 1 \leq v \leq m$, there exists a subdivision $(Q_i)_{i \in I}$ of Q and, for each v , a subset I_v of I such that $(Q_i)_{i \in I_v}$ is a subdivision of R_v .

Let $(\mathcal{R}, \text{div})$ be a rectangular system on \mathcal{X} . A complex valued function φ defined on \mathcal{R} is said to be *additive* if $\varphi(Q) = \sum_{i \in I} \varphi(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q . Similarly, a real-valued function s defined on \mathcal{R} is called *subadditive* if $s(Q) \leq \sum_{i \in I} s(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q . The function s is called *superadditive* if $-s$ is subadditive.

DEFINITION 3.3. Let φ be additive and s superadditive on \mathcal{R} . A subset $A \subset \mathcal{X}$ is said to be (φ, s) -negligible if, for any $Q \in \mathcal{R}$ and any $\varepsilon > 0$, there exist a subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I so that $Q_i \cap A = \emptyset$ for any $i \in I \setminus J$, and such that

$$\left| \sum_{i \in J} \varphi(Q_i) \right| \leq \sum_{i \in J} s(Q_i) + \varepsilon.$$

We are now in a position to state in precise terms the localization principle alluded to in the introduction.

THEOREM 3.4. Let $(\mathcal{R}, \text{div})$ be a rectangular system on the complete metric space \mathcal{X} . Also, let φ be an additive function on \mathcal{R} , s a superadditive function on \mathcal{R} , and let $A \subset \mathcal{X}$ be a countable union of (φ, s) -negligible subsets of \mathcal{X} . The following conditions are equivalent:

(1) $|\varphi(Q)| \leq s(Q)$ for all $Q \in \mathcal{R}$;

(2) there exists a positive, superadditive function t on \mathcal{R} so that $|\varphi(Q)| \leq s(Q)$ whenever $t(Q) = 0$ and such that, for any nested sequence of rectangles $(Q_v)_v$ having $t(Q_v) > 0$ for all v , and $\bigcap_v Q_v = \{a\}$ for some $a \in \mathcal{X} \setminus A$, we have

$$\liminf_v \frac{|\varphi(Q_v)| - s(Q_v)}{t(Q_v)} \leq 0;$$

(3) there exists a positive, superadditive function t on \mathcal{R} and, for any $a \in \mathcal{X} \setminus A$ and any $\varepsilon > 0$, an open neighborhood \mathcal{U} of a in \mathcal{X} such that

$$|\varphi(Q)| \leq s(Q) + \varepsilon t(Q),$$

for any rectangle Q included in \mathcal{U} and containing a .

Furthermore, the above conditions still remain equivalent with $|\varphi(\cdot)|$ replaced by φ .

Note the analogy of this result with the maximum principle from potential theory (the additive and subadditive functions correspond to the harmonic and subharmonic functions, respectively, whereas χ appears as a kind of ideal boundary of \mathcal{R}).

Let us also point out that for $t = \text{constant}$, Theorem 3.4 is essentially a principle for passing from *local* to *global*, while for $t \neq \text{constant}$ a principle for passing from *infinitesimal* to *global*.

Proof of Theorem 3.4. Obviously, (1) implies (2). Moreover, a straightforward reasoning by contradiction shows that any function t satisfying the hypothesis (2) will automatically do for (3).

We are therefore left with $(3) \Rightarrow (1)$. Once again, we shall reason by contradiction. To this effect, assume that there exists a rectangle Q such that $|\varphi(Q)| > s(Q)$. In particular, this implies that $t(Q) > 0$. Now fix $\varepsilon > 0$, small enough so that

$$|\varphi(Q)| - s(Q) > \varepsilon t(Q),$$

and set $\varepsilon_v := (2^{-1} + 3^{-1-v})\varepsilon$. Let $A = \cup_{v=0}^{\infty} A_v$, where A_v is a (φ, s) -negligible subset of \mathcal{X} for each $v \in \mathbf{N}$. In particular, since A_0 is (φ, s) -negligible, there exist a subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $Q_i \cap A_0 = \emptyset$, when $i \in I \setminus J$, and such that

$$(3.1) \quad \left| \sum_{i \in J} \varphi(Q_i) \right| \leq \sum_{i \in J} s(Q_i) + |\varphi(Q)| - s(Q) - \varepsilon t(Q).$$

Next we claim that we cannot have $|\varphi(Q_i)| \leq s(Q_i) + \varepsilon_0 t(Q_i)$ for all $i \in I \setminus J$. To prove the claim, we remark that since t is positive and superadditive, this would lead to

$$(3.2) \quad \begin{aligned} \sum_{i \in I \setminus J} |\varphi(Q_i)| &\leq \sum_{i \in I \setminus J} s(Q_i) + \varepsilon_0 \sum_{i \in I \setminus J} t(Q_i) \\ &\leq \sum_{i \in I \setminus J} s(Q_i) + \varepsilon_0 t(Q). \end{aligned}$$

In turn, since φ is additive and s superadditive, a simple combination of (3.1) and (3.2) would imply that $0 \leq \varepsilon_0 t(Q) - \varepsilon t(Q)$, which is a contradiction. Consequently, one can find an index $i_0 \in I \setminus J$ for which

$$|\varphi(Q_{i_0})| > s(Q_{i_0}) + \varepsilon_0 t(Q_{i_0}).$$

Because Q_{i_0} and A_0 are disjoint it follows that it is possible to find a rectangle $R_0 \subseteq Q$ which does not intersect A_0 , has $\text{diam}(R_0) \leq 1$, and such that

$$|\varphi(R_0)| > s(R_0) + \varepsilon_0 t(R_0).$$

Continuing this inductively, one can construct a sequence of nested rectangles $\{R_v\}_v$ such that R_v does not meet A_v , $\text{diam}(R_v) \leq 2^{-v}$, and

$$|\varphi(R_v)| > s(R_v) + \varepsilon_v t(R_v) \geq s(R_v) + \frac{\varepsilon}{2} t(R_v),$$

for any $v \in \mathbf{N}$. But then $\cap_v R_v = \{a\}$ for some $a \in \mathcal{X} \setminus A$ and this contradicts (3). The proof is finished. \square

We shall also use the following version of the Theorem 3.4.

THEOREM 3.5. *Let $\mathcal{X}, (\mathcal{R}, \text{div}), A, \varphi, s$ be as in Theorem 3.4 and assume that t is a positive, superadditive function on \mathcal{R} . Then, the following conditions are equivalent:*

(1) *for any relatively compact open subset Ω of \mathcal{X} there exists $M > 0$ such that $|\varphi(Q)| \leq s(Q) + Mt(Q)$ for all $Q \in \mathcal{R}$ with $Q \subseteq \Omega$;*

(2) *$|\varphi(Q)| \leq s(Q)$ whenever $t(Q) = 0$ and for any nested sequence of rectangles $\{Q_v\}_v$ having $t(Q_v) > 0$ for all v , and $\cap_v Q_v = \{a\}$ for some $a \in \mathcal{X} \setminus A$, we have*

$$\limsup_v \frac{|\varphi(Q_v)| - s(Q_v)}{t(Q_v)} < +\infty;$$

(3) *for any $a \in \mathcal{X} \setminus A$ there exist an open neighborhood \mathcal{U} of a in \mathcal{X} and $M > 0$ such that*

$$|\varphi(Q)| \leq s(Q) + Mt(Q),$$

for any rectangle Q included in \mathcal{U} and containing a .

The proof is quite similar to that of Theorem 3.4 and we omit it.