Hence, on account of the continuity of ρ, we see that $\int_Q d\mu = \int_{\partial Q} u$, for any Q. With this at hand and by once again using the hypothesis (4), we conclude that μ is absolutely continuous with respect to the n-dimensional Lebesgue measure λ_n. Therefore, if $f \in L^1(\Omega, \mathrm{loc})$ denotes the Radon-Nikodym-Lebesgue density of μ with respect to λ_n, we have that

$$\int_{\partial Q} u = \int_Q d\mu = \iint_Q f$$

for any $Q \in \mathscr{R}(\Omega)$. Using this, Theorem 1.3 finally implies that $du = f$ in the distribution sense on Ω, and this concludes the proof of the theorem. $\square$

REMARK 2.2. Inspection of the proof also shows that $\rho(K) = \int_K |f| \, d\lambda_n$ for any compact subset K of Ω, and that $|f| \leqslant g$ a.e. on Ω.

An *integrally Lipschitz* $(n-1)$-*form* in Ω is a locally $(n-1)$-integrable form u for which there exists $M > 0$ so that

$$\left| \int_{\partial Q} u \right| \leqslant M \lambda_n(Q)$$

for each $Q \in \mathscr{R}(\Omega)$. Note that any integrally Lipschitz $(n-1)$-form u in Ω satisfies the equivalent conditions in Theorem 1.3.

3. A MAXIMUM PRINCIPLE

The purpose of this section is to prove a localization theorem which plays a significant role in the sequel. Here, our approach is of an abstract nature.

Let $\mathscr{X}$ be a fixed metric space. In general, for an arbitrary set E, we shall denote by $\mathscr{F}(E)$ the collection of all finite families of subsets of E, and by $\mathscr{S}(E)$ the collection of all subsets of $\mathscr{F}(E)$.

DEFINITION 3.1. *A rectangular system on* $\mathscr{X}$ *is a subset* $\mathscr{R}$ *of* $\mathrm{comp}(\mathscr{X})$ *together with an application* $\mathrm{div}: \mathscr{R} \to \mathscr{S}(\mathscr{R})$ *satisfying the following:*

(1) If $Q \in \mathscr{R}$ *and* $(Q_i)_{i \in I} \in \mathrm{div}(Q)$, *then* $Q_i \subseteq Q$ *for any* $i \in I$;

(2) For any $Q \in \mathscr{R}$ *and any* $\varepsilon > 0$, *there exists* $(Q_i)_{i \in I} \in \mathrm{div}(Q)$ *so that* $\mathrm{diam}(Q_i) < \varepsilon$ *for every* $i \in I$.

The elements of $\mathscr{R}$ will be called *rectangles,* whereas the elements of $\operatorname{div}(Q)$, for $Q \in \mathscr{R}$, will be called the *subdivisions* of Q. Later, we shall also need the following.

DEFINITION 3.2. *A rectangular system* $(\mathscr{R}, \operatorname{div})$ *is said to be full if for any* $Q \in \mathscr{R}$ *and any* $R_1, \ldots, R_m \in \mathscr{R}$ *with* $R_\nu \subseteq Q, 1 \leqslant \nu \leqslant m$, *there exists a subdivision* $(Q_i)_{i \in I}$ *of* Q *and, for each* ν, *a subset* I_ν *of* I *such that* $(Q_i)_{i \in I_\nu}$ *is a subdivision of* R_ν.

Let $(\mathscr{R}, \operatorname{div})$ be a rectangular system on $\mathscr{X}$. A complex valued function φ defined on $\mathscr{R}$ is said to be *additive* if $\varphi(Q) = \sum_{i \in I} \varphi(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q. Similarly, a real-valued function s defined on $\mathscr{R}$ is called *subadditive* if $s(Q) \leqslant \sum_{i \in I} s(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q. The function s is called *superadditive* if $-s$ is subadditive.

DEFINITION 3.3. *Let* φ *be additive and* s *superadditive on* $\mathscr{R}$. *A subset* $A \subset \mathscr{X}$ *is said to be* (φ, s)-*negligible if, for any* $Q \in \mathscr{R}$ *and any* $\varepsilon > 0$, *there exist a subdivision* $(Q_i)_{i \in I}$ *of* Q *and a subset* J *of* I *so that* $Q_i \cap A = \emptyset$ *for any* $i \in I \setminus J$, *and such that*

$$\left| \sum_{i \in J} \varphi(Q_i) \right| \leqslant \sum_{i \in J} s(Q_i) + \varepsilon .$$

We are now in a position to state in precise terms the localization principle alluded to in the introduction.

THEOREM 3.4. *Let* $(\mathscr{R}, \operatorname{div})$ *be a rectangular system on the complete metric space* $\mathscr{X}$. *Also, let* φ *be an additive function on* $\mathscr{R}, s$ *a superadditive function on* $\mathscr{R}$, *and let* $A \subset \mathscr{X}$ *be a countable union of* (φ, s)-*negligible subsets of* $\mathscr{X}$. *The following conditions are equivalent:*

(1) $|\varphi(Q)| \leqslant s(Q)$ *for all* $Q \in \mathscr{R}$;

(2) there exists a positive, superadditive function t *on* $\mathscr{R}$ *so that* $|\varphi(Q)| \leqslant s(Q)$ *whenever* $t(Q) = 0$ *and such that, for any nested sequence of rectangles* $(Q_\nu)_\nu$ *having* $t(Q_\nu) > 0$ *for all* ν, *and* $\cap_\nu Q_\nu = \{a\}$ *for some* $a \in \mathscr{X} \setminus A$, *we have*

$$\liminf_{\nu} \frac{|\varphi(Q_\nu)| - s(Q_\nu)}{t(Q_\nu)} \leqslant 0 ;$$

(3) there exists a positive, superadditive function t *on* $\mathscr{R}$ *and, for any* $a \in \mathscr{X} \backslash A$ *and any* $\varepsilon > 0$, *an open neighborhood* $\mathscr{U}$ *of* a *in* $\mathscr{X}$ *such that*

$$|\varphi(Q)| \leqslant s(Q) + \varepsilon t(Q),$$

for any rectangle Q *included in* $\mathscr{U}$ *and containing* a.

Furthermore, the above conditions still remain equivalent with $|\varphi(\cdot)|$ *replaced by* φ.

Note the analogy of this result with the maximum principle from potential theory (the additive and subadditive functions correspond to the harmonic and subharmonic functions, respectively, whereas χ appears as a kind of ideal boundary of $\mathscr{R}$).

Let us also point out that for t = constant, Theorem 3.4 is essentially a principle for passing from *local* to *global*, while for $t \neq$ constant a principle for passing from *infinitesimal* to *global.*

Proof of Theorem 3.4. Obviously, (1) implies (2). Moreover, a straightforward reasoning by contradiction shows that any function t satisfying the hypothesis (2) will automatically do for (3).

We are therefore left with (3) $\Rightarrow$ (1). Once again, we shall reason by contradiction. To this effect, assume that there exists a rectangle Q such that $|\varphi(Q)| > s(Q)$. In particular, this implies that $t(Q) > 0$. Now fix $\varepsilon > 0$, small enough so that

$$|\varphi(Q)| - s(Q) > \varepsilon t(Q),$$

and set $\varepsilon_\nu := (2^{-1} + 3^{-1-\nu})\varepsilon$. Let $A = \cup_{\nu=0}^{\infty} A_\nu$, where A_ν is a (φ, s)-negligible subset of $\mathscr{X}$ for each $\nu \in \mathbf{N}$. In particular, since A_0 is (φ, s)-negligible, there exist a subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $Q_i \cap A_0 = \varnothing$, when $i \in I \backslash J$, and such that

$$\left| \sum_{i \in J} \varphi(Q_i) \right| \leqslant \sum_{i \in J} s(Q_i) + |\varphi(Q)| - s(Q) - \varepsilon t(Q). \tag{3.1}$$

Next we claim that we cannot have $|\varphi(Q_i)| \leqslant s(Q_i) + \varepsilon_0 t(Q_i)$ for all $i \in I \backslash J$. To prove the claim, we remark that since t is positive and superadditive, this would lead to

$$\begin{aligned} \sum_{i \in I\backslash J} |\varphi(Q_i)| &\leqslant \sum_{i \in I\backslash J} s(Q_i) + \varepsilon_0 \sum_{i \in I\backslash J} t(Q_i) \\ &\leqslant \sum_{i \in I\backslash J} s(Q_i) + \varepsilon_0 t(Q). \end{aligned} \tag{3.2}$$

In turn, since φ is additive and s superadditive, a simple combination of (3.1) and (3.2) would imply that $0 \leqslant \varepsilon_0 t(Q) - \varepsilon t(Q)$, which is a contradiction. Consequently, one can find an index $i_0 \in I \setminus J$ for which

$$|\varphi(Q_{i_0})| > s(Q_{i_0}) + \varepsilon_0 t(Q_{i_0}) \,.$$

Because Q_{i_0} and A_0 are disjoint it follows that it is possible to find a rectangle $R_0 \subseteq Q$ which does not intersect A_0, has $\operatorname{diam}(R_0) \leqslant 1$, and such that

$$|\varphi(R_0)| > s(R_0) + \varepsilon_0 t(R_0) \,.$$

Continuing this inductively, one can construct a sequence of nested rectangles $\{R_\nu\}_\nu$ such that R_ν does not meet A_ν, $\operatorname{diam}(R_\nu) \leqslant 2^{-\nu}$, and

$$|\varphi(R_\nu)| > s(R_\nu) + \varepsilon_\nu t(R_\nu) \geqslant s(R_\nu) + \frac{\varepsilon}{2} t(R_\nu) \,,$$

for any $\nu \in \mathbf{N}$. But then $\cap_\nu R_\nu = \{a\}$ for some $a \in \mathscr{X} \setminus A$ and this contradicts (3). The proof is finished. $\square$

We shall also use the following version of the Theorem 3.4.

THEOREM 3.5. *Let* $\mathscr{X}, (\mathscr{R}, \operatorname{div}), A, \varphi, s$ *be as in Theorem 3.4 and assume that* t *is a positive, superadditive function on* $\mathscr{R}$. *Then, the following conditions are equivalent:*

(1) for any relatively compact open subset Ω *of* $\mathscr{X}$ *there exists* $M > 0$ *such that* $|\varphi(Q)| \leqslant s(Q) + Mt(Q)$ *for all* $Q \in \mathscr{R}$ *with* $Q \subseteqq \Omega$;

(2) $|\varphi(Q)| \leqslant s(Q)$ *whenever* $t(Q) = 0$ *and for any nested sequence of rectangles* $\{Q_\nu\}_\nu$ *having* $t(Q_\nu) > 0$ *for all* ν, *and* $\cap_\nu Q_\nu = \{a\}$ *for some* $a \in \mathscr{X} \setminus A$, *we have*

$$\limsup_\nu \frac{|\varphi(Q_\nu)| - s(Q_\nu)}{t(Q_\nu)} < +\infty \,;$$

(3) for any $a \in \mathscr{X} \setminus A$ *there exist an open neighborhood* $\mathscr{U}$ *of* a *in* $\mathscr{X}$ *and* $M > 0$ *such that*

$$|\varphi(Q)| \leqslant s(Q) + Mt(Q) \,,$$

for any rectangle Q *included in* $\mathscr{U}$ *and containing* a.

The proof is quite similar to that of Theorem 3.4 and we omit it.