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Hence, on account of the continuity of ρ , we see that $\int_Q d\mu = \int_{\partial Q} u$, for any Q . With this at hand and by once again using the hypothesis (4), we conclude that μ is absolutely continuous with respect to the n -dimensional Lebesgue measure λ_n . Therefore, if $f \in L^1(\Omega, \text{loc})$ denotes the Radon-Nikodym-Lebesgue density of μ with respect to λ_n , we have that

$$\int_{\partial Q} u = \int_Q d\mu = \iint_Q f$$

for any $Q \in \mathcal{R}(\Omega)$. Using this, Theorem 1.3 finally implies that $du = f$ in the distribution sense on Ω , and this concludes the proof of the theorem. \square

REMARK 2.2. Inspection of the proof also shows that $\rho(K) = \int_K |f| d\lambda_n$ for any compact subset K of Ω , and that $|f| \leq g$ a.e. on Ω .

An *integrally Lipschitz* $(n - 1)$ -form in Ω is a locally $(n - 1)$ -integrable form u for which there exists $M > 0$ so that

$$\left| \int_{\partial Q} u \right| \leq M\lambda_n(Q)$$

for each $Q \in \mathcal{R}(\Omega)$. Note that any integrally Lipschitz $(n - 1)$ -form u in Ω satisfies the equivalent conditions in Theorem 1.3.

3. A MAXIMUM PRINCIPLE

The purpose of this section is to prove a localization theorem which plays a significant role in the sequel. Here, our approach is of an abstract nature.

Let \mathcal{X} be a fixed metric space. In general, for an arbitrary set E , we shall denote by $\mathcal{F}(E)$ the collection of all finite families of subsets of E , and by $\mathcal{S}(E)$ the collection of all subsets of $\mathcal{F}(E)$.

DEFINITION 3.1. A *rectangular system* on \mathcal{X} is a subset \mathcal{R} of $\text{comp}(\mathcal{X})$ together with an application $\text{div}: \mathcal{R} \rightarrow \mathcal{S}(\mathcal{R})$ satisfying the following:

- (1) If $Q \in \mathcal{R}$ and $(Q_i)_{i \in I} \in \text{div}(Q)$, then $Q_i \subseteq Q$ for any $i \in I$;
- (2) For any $Q \in \mathcal{R}$ and any $\varepsilon > 0$, there exists $(Q_i)_{i \in I} \in \text{div}(Q)$ so that $\text{diam}(Q_i) < \varepsilon$ for every $i \in I$.

The elements of \mathcal{R} will be called *rectangles*, whereas the elements of $\text{div}(Q)$, for $Q \in \mathcal{R}$, will be called the *subdivisions* of Q . Later, we shall also need the following.

DEFINITION 3.2. A rectangular system $(\mathcal{R}, \text{div})$ is said to be full if for any $Q \in \mathcal{R}$ and any $R_1, \dots, R_m \in \mathcal{R}$ with $R_\nu \subseteq Q, 1 \leq \nu \leq m$, there exists a subdivision $(Q_i)_{i \in I}$ of Q and, for each ν , a subset I_ν of I such that $(Q_i)_{i \in I_\nu}$ is a subdivision of R_ν .

Let $(\mathcal{R}, \text{div})$ be a rectangular system on \mathcal{X} . A complex valued function φ defined on \mathcal{R} is said to be *additive* if $\varphi(Q) = \sum_{i \in I} \varphi(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q . Similarly, a real-valued function s defined on \mathcal{R} is called *subadditive* if $s(Q) \leq \sum_{i \in I} s(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q . The function s is called *superadditive* if $-s$ is subadditive.

DEFINITION 3.3. Let φ be additive and s superadditive on \mathcal{R} . A subset $A \subset \mathcal{X}$ is said to be (φ, s) -negligible if, for any $Q \in \mathcal{R}$ and any $\varepsilon > 0$, there exist a subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I so that $Q_i \cap A = \emptyset$ for any $i \in I \setminus J$, and such that

$$\left| \sum_{i \in J} \varphi(Q_i) \right| \leq \sum_{i \in J} s(Q_i) + \varepsilon.$$

We are now in a position to state in precise terms the localization principle alluded to in the introduction.

THEOREM 3.4. Let $(\mathcal{R}, \text{div})$ be a rectangular system on the complete metric space \mathcal{X} . Also, let φ be an additive function on \mathcal{R} , s a superadditive function on \mathcal{R} , and let $A \subset \mathcal{X}$ be a countable union of (φ, s) -negligible subsets of \mathcal{X} . The following conditions are equivalent:

(1) $|\varphi(Q)| \leq s(Q)$ for all $Q \in \mathcal{R}$;

(2) there exists a positive, superadditive function t on \mathcal{R} so that $|\varphi(Q)| \leq s(Q)$ whenever $t(Q) = 0$ and such that, for any nested sequence of rectangles $(Q_\nu)_\nu$ having $t(Q_\nu) > 0$ for all ν , and $\bigcap_\nu Q_\nu = \{a\}$ for some $a \in \mathcal{X} \setminus A$, we have

$$\liminf_\nu \frac{|\varphi(Q_\nu)| - s(Q_\nu)}{t(Q_\nu)} \leq 0;$$

(3) there exists a positive, superadditive function t on \mathcal{R} and, for any $a \in \mathcal{X} \setminus A$ and any $\varepsilon > 0$, an open neighborhood \mathcal{U} of a in \mathcal{X} such that

$$|\varphi(Q)| \leq s(Q) + \varepsilon t(Q),$$

for any rectangle Q included in \mathcal{U} and containing a .

Furthermore, the above conditions still remain equivalent with $|\varphi(\cdot)|$ replaced by φ .

Note the analogy of this result with the maximum principle from potential theory (the additive and subadditive functions correspond to the harmonic and subharmonic functions, respectively, whereas χ appears as a kind of ideal boundary of \mathcal{R}).

Let us also point out that for $t = \text{constant}$, Theorem 3.4 is essentially a principle for passing from *local* to *global*, while for $t \neq \text{constant}$ a principle for passing from *infinitesimal* to *global*.

Proof of Theorem 3.4. Obviously, (1) implies (2). Moreover, a straightforward reasoning by contradiction shows that any function t satisfying the hypothesis (2) will automatically do for (3).

We are therefore left with (3) \Rightarrow (1). Once again, we shall reason by contradiction. To this effect, assume that there exists a rectangle Q such that $|\varphi(Q)| > s(Q)$. In particular, this implies that $t(Q) > 0$. Now fix $\varepsilon > 0$, small enough so that

$$|\varphi(Q)| - s(Q) > \varepsilon t(Q),$$

and set $\varepsilon_\nu := (2^{-1} + 3^{-1-\nu})\varepsilon$. Let $A = \cup_{\nu=0}^{\infty} A_\nu$, where A_ν is a (φ, s) -negligible subset of \mathcal{X} for each $\nu \in \mathbf{N}$. In particular, since A_0 is (φ, s) -negligible, there exist a subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $Q_i \cap A_0 = \emptyset$, when $i \in I \setminus J$, and such that

$$(3.1) \quad \left| \sum_{i \in J} \varphi(Q_i) \right| \leq \sum_{i \in J} s(Q_i) + |\varphi(Q)| - s(Q) - \varepsilon t(Q).$$

Next we claim that we cannot have $|\varphi(Q_i)| \leq s(Q_i) + \varepsilon_0 t(Q_i)$ for all $i \in I \setminus J$. To prove the claim, we remark that since t is positive and superadditive, this would lead to

$$(3.2) \quad \begin{aligned} \sum_{i \in I \setminus J} |\varphi(Q_i)| &\leq \sum_{i \in I \setminus J} s(Q_i) + \varepsilon_0 \sum_{i \in I \setminus J} t(Q_i) \\ &\leq \sum_{i \in I \setminus J} s(Q_i) + \varepsilon_0 t(Q). \end{aligned}$$

In turn, since φ is additive and s superadditive, a simple combination of (3.1) and (3.2) would imply that $0 \leq \varepsilon_0 t(Q) - \varepsilon t(Q)$, which is a contradiction. Consequently, one can find an index $i_0 \in I \setminus J$ for which

$$|\varphi(Q_{i_0})| > s(Q_{i_0}) + \varepsilon_0 t(Q_{i_0}).$$

Because Q_{i_0} and A_0 are disjoint it follows that it is possible to find a rectangle $R_0 \subseteq Q$ which does not intersect A_0 , has $\text{diam}(R_0) \leq 1$, and such that

$$|\varphi(R_0)| > s(R_0) + \varepsilon_0 t(R_0).$$

Continuing this inductively, one can construct a sequence of nested rectangles $\{R_\nu\}_\nu$ such that R_ν does not meet A_ν , $\text{diam}(R_\nu) \leq 2^{-\nu}$, and

$$|\varphi(R_\nu)| > s(R_\nu) + \varepsilon_\nu t(R_\nu) \geq s(R_\nu) + \frac{\varepsilon}{2} t(R_\nu),$$

for any $\nu \in \mathbf{N}$. But then $\bigcap_\nu R_\nu = \{a\}$ for some $a \in \mathcal{B} \setminus A$ and this contradicts (3). The proof is finished. \square

We shall also use the following version of the Theorem 3.4.

THEOREM 3.5. *Let $\mathcal{B}, (\mathcal{R}, \text{div}), A, \varphi, s$ be as in Theorem 3.4 and assume that t is a positive, superadditive function on \mathcal{R} . Then, the following conditions are equivalent:*

(1) *for any relatively compact open subset Ω of \mathcal{B} there exists $M > 0$ such that $|\varphi(Q)| \leq s(Q) + Mt(Q)$ for all $Q \in \mathcal{R}$ with $Q \subseteq \Omega$;*

(2) *$|\varphi(Q)| \leq s(Q)$ whenever $t(Q) = 0$ and for any nested sequence of rectangles $\{Q_\nu\}_\nu$ having $t(Q_\nu) > 0$ for all ν , and $\bigcap_\nu Q_\nu = \{a\}$ for some $a \in \mathcal{B} \setminus A$, we have*

$$\limsup_\nu \frac{|\varphi(Q_\nu)| - s(Q_\nu)}{t(Q_\nu)} < +\infty;$$

(3) *for any $a \in \mathcal{B} \setminus A$ there exist an open neighborhood \mathcal{U} of a in \mathcal{B} and $M > 0$ such that*

$$|\varphi(Q)| \leq s(Q) + Mt(Q),$$

for any rectangle Q included in \mathcal{U} and containing a .

The proof is quite similar to that of Theorem 3.4 and we omit it.