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Artikel: PLURIDIMENSIONAL ABSOLUTE CONTINUITY FOR DIFFERENTIAL FORMS AND THE STOKES FORMULA

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where the coefficients of the $(n-1)$-form v as well as $(w_i)_i$ are Lipschitz functions. Clearly, the usual Stokes formula on $[0,1]^n$ holds for v whereas, for $1 \leqslant i \leqslant n-1$,

$$\int_{[0,1]^{n-1}} (w_i(x',1) - w_i(x',0))\partial_i \varphi(x')dx'$$

$$= (-1)^{n-1} \int_0^1 \int_{[0,1]^{n-1}} \frac{\partial w_i(x',x_n)}{\partial x_n} \partial_i \varphi(x')dx' \wedge dx_n .$$

Consequently, the Stokes formula holds for h^*u on $[0,1]^n$, so that

$$\int_{\partial K} u = \int_{\partial[0,1]^n} h^*u = \iint_{[0,1]^n} d(h^*u) = \iint_{[0,1]^n} h^*(du)$$

$$= \iint_{[0,1]^n} h^*f = \iint_K f$$

and the proof is complete. □

DEFINITION 1.5. *Let* Ω *be a Lipschitz domain in* $\mathbf{R}^n$. *An integrally continuous* $(n-1)$-*form* u *on* Ω *is called absolutely continuous on* Ω *if* $d(u\,|_{\mathring{\Omega}})$, *taken in the distribution sense, is integrable on* $\mathring{K}$ *for any compact subset* K *of* Ω.

Note that if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and u_i are, for instance, locally Lipschitz on Ω, then u is absolutely continuous on Ω.

A simple consequence of Theorem 1.3 and of the above definition is the next.

THEOREM 1.6. *If* K *is a compact Lipschitz domain in* $\mathbf{R}^n$ *and* u *is an absolutely continuous* $(n-1)$-*form on* K, *then*

$$\int_{\partial K} u = \iint_{\mathring{K}} du .$$

2. CHARACTERIZATIONS OF THE PLURIDIMENSIONAL ABSOLUTE CONTINUITY

Theorem 1.3 suggests the possibility of characterizing pluridimensional absolute continuity of $(n-1)$-forms in a way similar to Lebesgue's definition of absolute continuity of functions on the real line (i.e. without involving the exterior derivative operator). This is made precise in the following theorem.

THEOREM 2.1. *Let* Ω *be an open subset of* $\mathbf{R}^n$ *and let* u *be a* $(n-1)$*-form which is locally* $(n-1)$*-integrable on* Ω. *The following are equivalent.*

(1) There exists a locally integrable n-form f *on* Ω *such that* $du = f$ *in the distribution sense on* Ω.

(2) There exists a locally integrable n-form f *on* Ω *such that* $\int_{\partial Q} u = \iint_Q f$, *for any* $Q \in \mathscr{R}(\Omega)$.

(3) There exists a locally integrable n-form g *on* Ω *such that* $\left| \int_{\partial Q} u \right| \leqslant \iint_Q g$, *for any* $Q \in \mathscr{R}(\Omega)$.

(4) For any $Q \in \mathscr{R}(\Omega)$ *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\sum_{i \in J} \left| \int_{\partial Q_i} u \right| \leqslant \varepsilon ,$$

for any subdivision $(Q_i)_{i \in I}$ *of* Q *and any* $J \subseteq I$ *for which* $\sum_{i \in J} \lambda_n(Q_i) \leqslant \delta$.

In particular, Theorem 1.3 and the above result show that an integrally continuous $(n-1)$-form u on Ω is absolutely continuous on Ω if it satisfies one of the above equivalent conditions. However, let us note that, without the integral continuity condition, u with (1)-(4) above is not necessarily absolutely continuous, except for $n = 1$.

Here is a simple counterexample in $\mathbf{R}^2$. If χ is the characteristic function of $\{(1-t, t); 0 \leqslant t \leqslant 1\} \subset \mathbf{R}^2$, then $u := \chi \, dx_2$ is locally 1-integrable and satisfies (1) – (4) in the above theorem, without being absolutely continuous on $\Omega := \mathbf{R}^2$.

Proof of Theorem 2.1. Clearly, all we need to show is that (4) implies (1). For each rectangle Q contained in Ω we set

$$\rho(Q) := \sup \left\{ \sum_{i \in I} \left| \int_{\partial Q_i} u \right| ; (Q_i)_{i \in I} \text{ an elementary subdivision of } Q \right\}. \tag{2.1}$$

Note that $\left| \int_{\partial Q} u \right| \leqslant \rho(Q) < +\infty$ for any rectangle Q. Also, since $Q \mapsto \int_{\partial Q} u$ is *rectangle-additive,* i.e. $\int_{\partial Q} u = \sum_{i \in I} \int_{\partial Q_i} u$ for any rectangle Q and any subdivision $(Q_i)_{i \in I}$ of Q, so is ρ. Therefore, it makes sense to extend ρ by setting

$$\rho(P) := \sum_{i \in I} \rho(Q_i) , \tag{2.2}$$

for any paved set P contained in Ω and any subdivision $(Q_i)_{i \in I}$ of P. The rectangle-additivity of ρ ensures that this extension is consistent with (2.1) and that (2.2) is independent of the particular choice of the subdivision $(Q_i)_{i \in I}$ of P. Going further, we extend ρ to $\mathrm{comp}(\Omega)$ by setting

$$\rho(K) := \inf\{\rho(P);\ P \text{ paved set, } K \subseteq P\}, \quad K \in \mathrm{comp}(\Omega) .$$

By (4), this extension is continuous in the sense that $\rho(K_\nu) \to \rho(K)$ whenever $(K_\nu)_\nu$ is a nested sequence of compact sets in Ω such that $\cap_\nu K_\nu = K$.

Now, for each multi-index $\alpha \in \mathbf{N}^n$ and for each $k \in \mathbf{N}$ we consider the cube $Q_{k,\alpha} := [0, 2^{-k}]^n + 2^{-k}\alpha$, and the set of multi-indices $I_k := \{\alpha \in \mathbf{N}^n;\ Q_{k,\alpha} \subseteq \Omega\}$. Moreover, for any complex-valued, continuous and compactly supported function ψ on Ω, we set

$$I_k(\psi) := \{\alpha \in I_k;\ \mathrm{supp}\,\psi \cap Q_{k,\alpha} \neq \varnothing\}$$

and

$$P_k(\psi) := \bigcup_{\alpha \in I_k(\psi)} Q_{k,\alpha} .$$

It follows that $P_{k+1}(\psi) \subseteq P_k(\psi)$ for any $k \in \mathbf{N}$ and that $\cap_k P_k(\psi) = \mathrm{supp}\,\psi$.

Next, we define

$$s_k(\psi) := \sum_{\alpha \in I_k(\psi)} \psi(2^{-k}\alpha) \int_{\partial Q_{k,\alpha}} u .$$

Clearly, s_k is a $\mathbf{C}$-linear functional on $C_0(\Omega)$ which satisfies

$$|s_k(\psi)| \leqslant \rho(P_k(\psi)) \sup_\Omega |\psi|, \quad \psi \in C_0(\Omega) .$$

Finally, we introduce $\mu : C_0(\Omega) \to \mathbf{C}$ by setting

$$\mu(\psi) := \lim_k s_k(\psi), \quad \psi \in C_0(\Omega) ,$$

where the existence of the limit easily follows from the uniform continuity of ψ. As μ is $\mathbf{C}$-linear and satisfies $|\mu(\psi)| \leqslant \rho(\mathrm{supp}\,\psi)\, \|\psi\|_{L^\infty}$, we infer that μ is a complex-valued Radon measure on Ω.

Fix $Q \in \mathscr{R}(\Omega)$ and take $\psi_\nu \in C_0(\Omega)$ a sequence of real-valued functions such that $0 \leqslant \psi_\nu \leqslant 1$ on Ω, $\psi_\nu = 1$ on a neighborhood of Q, $\mathrm{supp}\,\psi_{\nu+1} \subseteq \mathrm{supp}\,\psi_\nu$ and $\cap_\nu \mathrm{supp}\,\psi_\nu = Q$. From the definition of μ it is not difficult to see that

$$\left| \mu(\psi_\nu) - \int_{\partial Q} u \right| \leqslant \rho(\mathrm{supp}\,\psi_\nu) - \rho(Q) .$$

Hence, on account of the continuity of ρ, we see that $\int_Q d\mu = \int_{\partial Q} u$, for any Q. With this at hand and by once again using the hypothesis (4), we conclude that μ is absolutely continuous with respect to the n-dimensional Lebesgue measure λ_n. Therefore, if $f \in L^1(\Omega, \mathrm{loc})$ denotes the Radon-Nikodym-Lebesgue density of μ with respect to λ_n, we have that

$$\int_{\partial Q} u = \int_Q d\mu = \iint_Q f$$

for any $Q \in \mathscr{R}(\Omega)$. Using this, Theorem 1.3 finally implies that $du = f$ in the distribution sense on Ω, and this concludes the proof of the theorem. □

REMARK 2.2. Inspection of the proof also shows that $\rho(K) = \int_K |f| \, d\lambda_n$ for any compact subset K of Ω, and that $|f| \leqslant g$ a.e. on Ω.

An *integrally Lipschitz* $(n-1)$-*form* in Ω is a locally $(n-1)$-integrable form u for which there exists $M > 0$ so that

$$\left| \int_{\partial Q} u \right| \leqslant M \lambda_n(Q)$$

for each $Q \in \mathscr{R}(\Omega)$. Note that any integrally Lipschitz $(n-1)$-form u in Ω satisfies the equivalent conditions in Theorem 1.3.

3. A MAXIMUM PRINCIPLE

The purpose of this section is to prove a localization theorem which plays a significant role in the sequel. Here, our approach is of an abstract nature.

Let $\mathscr{X}$ be a fixed metric space. In general, for an arbitrary set E, we shall denote by $\mathscr{F}(E)$ the collection of all finite families of subsets of E, and by $\mathscr{S}(E)$ the collection of all subsets of $\mathscr{F}(E)$.

DEFINITION 3.1. *A rectangular system on* $\mathscr{X}$ *is a subset* $\mathscr{R}$ *of* $\mathrm{comp}(\mathscr{X})$ *together with an application* $\mathrm{div}: \mathscr{R} \to \mathscr{S}(\mathscr{R})$ *satisfying the following:*

(1) If $Q \in \mathscr{R}$ *and* $(Q_i)_{i \in I} \in \mathrm{div}(Q)$, *then* $Q_i \subseteq Q$ *for any* $i \in I$;

(2) For any $Q \in \mathscr{R}$ *and any* $\varepsilon > 0$, *there exists* $(Q_i)_{i \in I} \in \mathrm{div}(Q)$ *so that* $\mathrm{diam}(Q_i) < \varepsilon$ *for every* $i \in I$.