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1. INTEGRAL AND ABSOLUTE CONTINUITY AND THE LOCAL PROBLEM

DEFINITION 1.1. A bounded subset Ω of \mathbf{R}^n is called a *Lipschitz domain* if for any $a \in \Omega \setminus \overset{\circ}{\Omega}$, there exists an open neighborhood U of a in \mathbf{R}^n , a coordinate system (isometric to the canonical one) $(x', x_n) = ((x_1, \dots, x_{n-1}), x_n)$, and a Lipschitz continuous function $\varphi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ such that

$$\Omega \cap U = \{(x', x_n); \varphi(x') \leq x_n\} \cap U.$$

Also, if the new coordinates are actually obtained by permuting the canonical ones, then Ω is called a *simple Lipschitz domain*.

Note that, the *border* of the domain Ω , $b\Omega := \Omega \setminus \overset{\circ}{\Omega}$, is either the empty set or a $(n-1)$ -dimensional Lipschitz submanifold of \mathbf{R}^n (assumed with the standard induced orientation).

Let now Ω be a Lipschitz domain in \mathbf{R}^n and ω an open set in \mathbf{R}^{n-1} . A locally bi-Lipschitz mapping $\varphi: \omega \rightarrow \Omega$ is called *Lipschitz embedding* provided φ maps ω homeomorphically onto $\varphi(\omega)$. Furthermore, if S is a topological space, $h: S \times \omega \rightarrow \Omega$ is called a *continuous family of Lipschitz embeddings* if $h_s := h(s, \cdot)$ is a Lipschitz embedding for each fixed $s \in S$, and if the mappings

$$(1.1) \quad S \ni s \mapsto \frac{\partial h_s}{\partial x_i} \in L^\infty(\omega, \text{loc}), \quad i = 1, \dots, n-1,$$

are continuous. Here $L^\infty(\omega, \text{loc})$ is endowed with the usual (Fréchet) topology given by uniform convergence on compact subsets of ω . Throughout this paper S will actually always be a locally closed subspace of some \mathbf{R}^k .

Let $L^1(\Omega, \text{loc})$ stand for the vector space of differential forms with locally integrable coefficients on Ω . We consider this space endowed with the usual (locally convex) topology.

DEFINITION 1.2. A complex-valued $(n-1)$ -form u defined on Ω is called *integrally continuous* if:

- (1) the form u is locally $(n-1)$ -integrable, i.e. φ^*u is locally integrable on ω for any Lipschitz embedding $\varphi: \omega \rightarrow \Omega$;
- (2) the mapping $S \ni s \mapsto h_s^*u \in L^1(\omega, \text{loc})$ is continuous, for any continuous family of Lipschitz embeddings $h: S \times \omega \rightarrow \Omega$.

EXAMPLES. Let $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$ be a $(n-1)$ -form on Ω where, as usual, the symbol under the “hat” is omitted in the product.

(1) If for $1 \leq i \leq n$ the functions $u_i|_C$ are μ_{n-1} -integrable for any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$, then u is locally $(n-1)$ -integrable. In particular, this is the case if u_i are locally bounded.

(2) Suppose that there exists a set $A \subset \Omega$ of zero $(n-1)$ -dimensional Hausdorff measure such that $u_i|_{(\Omega \setminus A)}$ is continuous (in the induced topology) and $u_i|_{C \cap (\Omega \setminus A)}$ is μ_{n-1} integrable for any $1 \leq i \leq n$ and any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$. Then u is integrally continuous as well.

(3) For $n = 1$, a $(n-1)$ -form u is a function and, in this case, the form u is integrally continuous if and only if the function u is continuous.

Recall the usual exterior derivative operator d . The main result of this section is the following.

THEOREM 1.3. *Consider a Lipschitz domain Ω in \mathbf{R}^n . Let u be an integrally continuous $(n-1)$ -form on Ω and let f be a locally integrable n -form on Ω . The following are equivalent.*

- (1) *For any compact Lipschitz domain $K \subseteq \Omega$ we have $\int_{\partial K} u = \iint_K f$.*
- (2) *For any rectangle $Q \in \mathcal{R}(\Omega)$ we have $\int_{\partial Q} u = \iint_Q f$.*
- (3) *$du = f$ in the distribution sense on $\overset{\circ}{\Omega}$.*

Before we proceed with the proof of this theorem, we shall prove a lemma. To state it, we need some more notation. Let χ be a positive, smooth, function supported in the closed unit ball in \mathbf{R}^n and normalized such that $\iint_{\mathbf{R}^n} \chi dx = 1$. For $\varepsilon > 0$, set $\Omega_\varepsilon := \{x \in \mathbf{R}^n; \text{dist}(x, \partial\Omega) > \varepsilon\}$ and, for any $\Phi \in L^1(\overset{\circ}{\Omega}, \text{loc})$, set

$$\Phi_\varepsilon(x) := \iint_{\mathbf{R}^n} \Phi(x - \varepsilon y) \chi(y) dy, \quad x \in \Omega_\varepsilon.$$

It is a well-known fact that $\Phi_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and that $\Phi_\varepsilon \rightarrow \Phi$ in $L^1(\overset{\circ}{\Omega}, \text{loc})$ as ε tends to zero. For a locally integrable form u on Ω , u_ε is defined componentwise.

LEMMA 1.4. *Let Ω be a Lipschitz domain in \mathbf{R}^n and let u be an integrally continuous $(n-1)$ -form on Ω . Then:*

- (1) $u \in L^1(\Omega, \text{loc})$;
- (2) $\varphi^* u_\varepsilon \rightarrow \varphi^* u$ in $L^1(\omega, \text{loc})$ as ε approaches zero, for any Lipschitz embedding $\varphi: \omega \rightarrow \overset{\circ}{\Omega}$.

Proof. For each sufficiently small $\varepsilon > 0$, fixed for the moment, consider the continuous family of Lipschitz embeddings

$$h: \Omega_\varepsilon \times \omega_\varepsilon \ni (x, t) \mapsto (x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n),$$

where Ω_ε , defined above, stands for the space of parameters and ω_ε stands for a suitably small, open neighborhood of the cube $[-\varepsilon, \varepsilon]^{n-1}$. Obviously, $(h_x^* u)(t) = \Phi(x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \dots \wedge dt_{n-1}$, for some function Φ . Since u is integrally continuous, the function

$$\Phi^\varepsilon(x) := \varepsilon^{1-n} \int_{[-\varepsilon, \varepsilon]^{n-1}} \Phi(x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \dots \wedge dt_{n-1}$$

is continuous on Ω_ε . For any small, fixed x_n , the Lebesgue differentiation theorem yields that $\Phi^\varepsilon(\cdot, x_n) \rightarrow \Phi(\cdot, x_n)$, as $\varepsilon \rightarrow 0$, almost everywhere with respect to the $(n-1)$ -dimensional Lebesgue measure on $\{x' \in \mathbf{R}^{n-1}; (x', x_n) \in \mathring{\Omega}\}$. Using Fubini's theorem we infer that $\Phi^\varepsilon \rightarrow \Phi$, as $\varepsilon \rightarrow 0$, almost everywhere on Ω . Thus, Φ is λ_n -measurable.

Next, let $Q = Q' \times Q_n$ be a rectangle in $\mathbf{R}^{n-1} \times \mathbf{R}$ which is contained in $\mathring{\Omega}$, and consider the continuous family of Lipschitz embeddings

$$k: Q_n \times Q' \ni (x_n, x') \mapsto (x', x_n) \in \Omega.$$

Hence, $k_{x_n}^* u = \Phi(\cdot, x_n) dx_1 \wedge \dots \wedge dx_{n-1}$. As u is integrally continuous, the mapping

$$Q_n \ni x_n \mapsto \int_{Q'} |\Phi(x', x_n)| dx_1 \wedge \dots \wedge dx_{n-1}$$

is continuous. In particular, the iterated integral

$$\int_{Q_n} \int_{Q'} |\Phi(x', x_n)| dx' \wedge dx_n$$

is finite. By Fubini's theorem, it follows that Φ is integrable on Q .

Now, if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$ the above reasoning gives that $u_1 = \Phi$ is integrable on Q . Likewise, u_2, \dots, u_n are integrable on Q , and u is thus locally integrable on $\mathring{\Omega}$.

To conclude the proof of (1) it suffices to show that any $a \in \Omega \setminus \mathring{\Omega}$ has a compact neighborhood K in \mathbf{R}^n such that u is integrable on $K \cap \Omega$. To see this, there is no loss of generality assuming that K is so that

$$K \cap \Omega = \{(x', x_n); x' \in Q', \varphi(x') \leq x_n \leq \varphi(x') + \varepsilon\},$$

where Q' is a rectangle in \mathbf{R}^{n-1} , $\varphi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ is a Lipschitz function, and $\varepsilon > 0$ is some fixed, sufficiently small number. This time we take the continuous family of Lipschitz embeddings

$$h': [0, \varepsilon] \times Q' \ni (s, x') \mapsto (x', \varphi(x') + s) \in \Omega$$

and proceed as before. Hence, (1) follows.

To see (2), let $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ in Ω , so that we have $\varphi^* u = (\sum_i (u_i \circ \varphi) \Phi_i) dt_1 \wedge \cdots \wedge dt_{n-1}$, where Φ_i are measurable functions, locally (essentially) bounded on ω . Similarly, $\varphi^* u_\varepsilon = (\sum_i ((u_i)_\varepsilon \circ \varphi) \Phi_i) dt_1 \wedge \cdots \wedge dt_{n-1}$.

Given a compact subset C of ω , we consider the continuous family of Lipschitz embeddings $(s, t) \mapsto \varphi(t) - s$, where t lies in an open neighborhood of C and s lies in a small open ball centered at the origin of \mathbf{R}^n . Let also $\theta > 0$ be an arbitrary, fixed number. By the integral continuity of u , there exists $\delta > 0$ such that

$$(1.2) \quad \int_C \left| \sum_{i=1}^n u_i(\varphi(t) - \varepsilon y) \Phi_i(t) - \sum_{i=1}^n u_i(\varphi(t)) \Phi_i(t) \right| \chi(y) dt < \theta$$

for all $|y| \leq 1$ and $0 < \varepsilon < \delta$. Since, by (1), the functions u_i are locally integrable on $\mathring{\Omega}$, the function

$$\varepsilon^{-n} \sum_i u_i(y) \Phi_i(t) \chi\left(\frac{\varphi(t) - y}{\varepsilon}\right), \quad \varepsilon > 0,$$

is locally integrable on $\omega \times \mathring{\Omega}$. Integrating (1.2) against dy over the closed unit ball in \mathbf{R}^n and then changing the order of integration, we obtain

$$\int_C |\varphi^* u_\varepsilon - \varphi^* u| d\mu_{n-1} \leq c_n \theta,$$

for some $c_n > 0$ depending only on n . Since $\theta > 0$ was arbitrary, the proof of the lemma is therefore complete. \square

Proof of Theorem 1.3. Obviously (1) implies (2). Next, assume that (2) holds and let Q be an arbitrary rectangle in \mathbf{R}^n contained in $\mathring{\Omega}$. It is then straightforward to see that, for a sufficiently small $\varepsilon > 0$,

$$\int_{\partial Q} u_\varepsilon = \iint_Q f_\varepsilon.$$

Since u_ε is smooth, the standard form of Stokes formula gives that $\iint_Q du_\varepsilon = \iint_Q f_\varepsilon$. As Q was arbitrarily chosen, we see that $du_\varepsilon = f_\varepsilon$ on Ω_ε and, hence, by letting ε go to zero, $du = f$ in the distribution sense on $\overset{\circ}{\Omega}$. Thus, (2) \Rightarrow (3).

Finally, we consider the implication (3) \Rightarrow (1). Using a smooth partition of unity, it is not difficult to see that matters can be reduced to verifying (1.1) in the following cases:

- (i) the support of u is included in the interior of K ;
- (ii) the domain K has the form

$$(1.3) \quad \{(x', x_n) \in [0, 1]^{n-1} \times [0, 1]; x_n \leq \varphi(x')\},$$

for some Lipschitz function $\varphi: \mathbf{R}^{n-1} \rightarrow (0, 1)$.

We present the proof in the second case, as the proof the first case goes along the same lines and is somewhat simpler. Let us first note that, if K_ε is as in (1.3) except that φ has been replaced by $\varepsilon\varphi$, with $0 < \varepsilon < 1$, on account of the integral continuity of u we have

$$\int_{\partial K} u = \lim_{\varepsilon \rightarrow 1} \int_{\partial K_\varepsilon} u.$$

Hence, it suffices to prove the statement with K_ε in place of K or, in other words, assuming that the compact domain K from (1.2) is actually contained in $\overset{\circ}{\Omega}$. Furthermore, since by (1) $du_\varepsilon = f_\varepsilon$ on Ω_ε for all $\varepsilon > 0$, and since

$$\int_{\partial K} u_\varepsilon \rightarrow \int_{\partial K} u, \quad \iint_K f_\varepsilon \rightarrow \iint_K f$$

as $\varepsilon \rightarrow 0$ (the first convergence utilizes the integral continuity of u), there is no loss in generality if we assume that u and f are smooth forms in a neighborhood of K .

Consider now the bi-Lipschitz homeomorphism

$$h: [0, 1]^n \ni (x', x_n) \mapsto (x', x_n \varphi(x')) \in K.$$

From the change of variable formula ([Fe3], Theorem 3.2.3, p. 243) we have

$$\int_{\partial K} u = \int_{\partial[0, 1]^n} h^* u.$$

Also, a routine calculation shows that

$$(h^* u)(x', x_n) = v(x', x_n) + \left(\sum_{i=1}^{n-1} w_i(x', x_n) \partial_i \varphi(x') \right) dx_1 \wedge \cdots \wedge dx_{n-1},$$

where the coefficients of the $(n-1)$ -form v as well as $(w_i)_i$ are Lipschitz functions. Clearly, the usual Stokes formula on $[0, 1]^n$ holds for v whereas, for $1 \leq i \leq n-1$,

$$\begin{aligned} & \int_{[0, 1]^{n-1}} (w_i(x', 1) - w_i(x', 0)) \partial_i \varphi(x') dx' \\ &= (-1)^{n-1} \int_0^1 \int_{[0, 1]^{n-1}} \frac{\partial w_i(x', x_n)}{\partial x_n} \partial_i \varphi(x') dx' \wedge dx_n. \end{aligned}$$

Consequently, the Stokes formula holds for h^*u on $[0, 1]^n$, so that

$$\begin{aligned} \int_{\partial K} u &= \int_{\partial [0, 1]^n} h^*u = \iint_{[0, 1]^n} d(h^*u) = \iint_{[0, 1]^n} h^*(du) \\ &= \iint_{[0, 1]^n} h^*f = \iint_K f \end{aligned}$$

and the proof is complete. \square

DEFINITION 1.5. Let Ω be a Lipschitz domain in \mathbf{R}^n . An integrally continuous $(n-1)$ -form u on Ω is called absolutely continuous on Ω if $d(u|_{\mathring{\Omega}})$, taken in the distribution sense, is integrable on \mathring{K} for any compact subset K of Ω .

Note that if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and u_i are, for instance, locally Lipschitz on Ω , then u is absolutely continuous on Ω .

A simple consequence of Theorem 1.3 and of the above definition is the next.

THEOREM 1.6. If K is a compact Lipschitz domain in \mathbf{R}^n and u is an absolutely continuous $(n-1)$ -form on K , then

$$\int_{\partial K} u = \iint_{\mathring{K}} du.$$

2. CHARACTERIZATIONS OF THE PLURIDIMENSIONAL ABSOLUTE CONTINUITY

Theorem 1.3 suggests the possibility of characterizing pluridimensional absolute continuity of $(n-1)$ -forms in a way similar to Lebesgue's definition of absolute continuity of functions on the real line (i.e. without involving the exterior derivative operator). This is made precise in the following theorem.