

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 41 (1995)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S  
THEOREM ON AUTOMORPHISMS OF THE BALL

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**Bibliographie**

**DOI:** <https://doi.org/10.5169/seals-61825>

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sion  $2n - 1$  in  $\mathcal{M}_{B_n}$ , it follows that  $\psi \circ (f \times g)$  vanishes on the connected component of  $\Omega \cap \mathcal{M}_{B_n}$  containing  $(z_0, \bar{z}_0)$ . After shrinking  $U$  if necessary, we can assume that  $\psi \circ (f \times g)$  vanishes on  $\Omega \cap \mathcal{M}_{B_n}$  and thus  $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$ . We consider the embedding  $\iota \times \iota: \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}_{\mathbf{C}}^n \times \mathbf{P}_{\mathbf{C}}^n$  given by  $\iota(z_1, \dots, z_n) = (\sqrt{-1}: z_1: \dots: z_n)$ , which maps  $\mathcal{M}_{B_n}$  onto a (dense open) subset of  $\mathcal{M}_{\mathbf{C}}^n$ . By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1}: \iota(U) \rightarrow \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1}: \iota(V) \rightarrow \iota(\hat{V}),$$

there exists  $A \in \text{PGL}(n+1, \mathbf{C})$  such that  $\tilde{f} = A|_{\iota(U)}$ . Thus  $f$  extends to the fractional linear map  $\iota^{-1} \circ A \circ \iota$ , which gives an automorphism of  $B_n$ .

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map  $f$  must be nonsingular at some point of  $U \cap \partial B_n$ . We begin by observing that  $f^{-1}(\partial B_n)$  is nowhere dense. Indeed, suppose on the contrary that  $f^{-1}(\partial B_n)$  contains a connected open set  $U_0$  and assume without loss of generality that  $f(z_0) = (1, 0, \dots, 0)$  for some point  $z_0 \in U_0$ . Then by the maximum principle,  $f_1 \equiv 1$  and hence  $f \equiv (1, 0, \dots, 0)$  on  $U_0$  and thus on  $U$ , contradicting the assumption that  $f$  is nonconstant. Now suppose on the contrary that the Jacobian determinant of  $f$  vanishes identically on  $U \cap \partial B_n$ . Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on  $U$ . As a consequence, the fibers of  $f$  contain no isolated points. Assume without loss of generality that  $(1, 0, \dots, 0) \in U$  and choose  $r < 1$  such that the spherical cap  $W := \{z \in B_n: \text{Re } z_1 > r\}$  is contained in  $U$ . Choose a point  $p \in W$  such that  $f(p) \notin \partial B_n$ . Let  $A$  be the connected component of  $f^{-1}(f(p)) \cap W$  that contains  $p$ ;  $A$  is an analytic subvariety of  $W$  of positive dimension. Furthermore  $\bar{A} \setminus A \subset \{z \in \mathbf{C}^n: \text{Re } z_1 = r\}$ . By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function  $\varphi: A \rightarrow \mathbf{C}$  given by  $\varphi(z) = \exp z_1$ , we conclude that  $\varphi$  is constant and thus  $\bar{A} \setminus A = \emptyset$  so that  $A$  is a compact subvariety of  $W$  of positive dimension, which is impossible.  $\square$

#### REFERENCES

- [Al] ALEXANDER, H. Holomorphic mappings from the ball and polydisc. *Math. Ann.* 209 (1974), 249-256.  
 [Ar] ARTIN, E. *Geometric Algebra*. Interscience Publishers, New York, 1957.  
 [BB] BLASCHKE, W. and G. BOL. *Geometrie der Gewebe*. Springer, Berlin, 1938.  
 [Ca] CARTAN, E. Sur le groupe de la géométrie hypersphérique. *Comm. Math. Helv.* 4 (1932), 158-171.

- [CG] CHERN, S.-S. and P. GRIFFITHS. Abel's theorem and webs. *Jahresber. Deutsch. Math.-Verein.* 80 (1978), 13-110. Corrections and addenda. *Jahresber. Deutsch. Math.-Verein.* 83 (1981), 78-83.
- [CJ] CHERN, S.-S. and S. JI. Projective geometry and Riemann's mapping problem. Preprint, 1994.
- [Co] COXETER, H.S.M. *Projective Geometry*. University of Toronto Press, Toronto, 1974.
- [Fo] FORSTNERIČ, F. Proper holomorphic mappings: a survey. *Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987-1988* (J.E. Fornæss, ed.). Princeton Univ. Press, Princeton, 1993, 297-363.
- [Go] GOLDBERG, V. *Theory of Multicodimensional  $(n + 1)$ -Webs*. Kluwer, Dordrecht, 1988.
- [Gu] GUNNING, R.C. *Introduction to Holomorphic Functions of Several Variables, Volume II: Local Theory*. Brooks/Cole, Pacific Grove, CA, 1990.
- [MY] MOK, N. and S.K. YEUNG. Geometric realizations of uniformization of conjugates of hermitian locally symmetric manifolds. *Complex Analysis and Geometry* (V. Ancona and A. Silva, eds.). Plenum Press, New York, 1993, 253-270.
- [MM] MOLZON, R. and K.P. MORTENSEN. The Schwarzian derivative for maps between manifolds with complex projective connections. Preprint, 1994.
- [Pe] PELLER, D. Proper holomorphic self-maps of the unit ball. *Math. Ann.* 190 (1971), 298-305. Correction. *Math. Ann.* 202 (1973), 135-136.
- [Po] POINCARÉ, H. Les fonctions analytiques de deux variables et la représentation conforme. *Rend. Circ. Mat. Palermo* (1907), 185-220.
- [Pr] PRENOWITZ, W. The characterization of plane collineations in terms of homologous families of lines. *Trans. Amer. Math. Soc.* 38 (1935), 564-599.
- [Ra] RADÓ, F. Non-injective collineations on some sets in Desarguesian projective planes and extension of non-commutative valuations. *Aequationes Math.* 4 (1970), 307-321.
- [Re] REIDEMEISTER, K. Topologische Fragen der Differentialgeometrie, V. *Math. Z.* 29 (1929), 427-435.
- [Ru] RUDIN, W. *Function Theory in the Unit Ball of  $C^n$* . Springer-Verlag, New York, 1980.
- [Ta] TANAKA, N. On pseudo-conformal geometry of hypersurfaces of the space of  $n$  complex variables. *J. Math. Soc. Japan* 14 (1962), 397-429.
- [We] WEBSTER, S. On the mapping problem for algebraic real hypersurfaces. *Inventiones Math.* 43 (1977), 53-68.

(Reçu le 5 août 1994)

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