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let  $G = \{e, \tau\} \cdot \text{PGL}(n+1, \mathbf{C})$ , where  $\tau: \mathbf{P}_{\mathbf{C}}^n \rightarrow \mathbf{P}_{\mathbf{C}}^n$  is given by  $\tau(z) = \bar{z}$  and  $e$  is the identity map. By Lemmas 5 and 4 applied to the restrictions  $f|_{U_j}$ , there are transformations  $A_j \in G$  such that  $f|_{U_j} = A_j|_{U_j}$ . Since an element of  $G$  is uniquely determined by its values on a nonempty open subset of  $\mathbf{P}_{\mathbf{C}}^n$  and  $(U_1 \cup \cdots \cup U_j) \cap U_{j+1} \neq \emptyset$ , it follows by induction that  $A_j = A_1$  for all  $j$ . Hence  $f = A_1|_U$ .  $\square$

### 3. THE POINCARÉ-TANAKA AND CHERN-JI THEOREMS

The Segre family  $\mathcal{M}_{B_n}$  mentioned in the introduction has the projective analogue

$$\mathcal{M}_K^n = \{(z, w) \in \mathbf{P}_K^n \times \mathbf{P}_K^n : \sum_{j=0}^n z_j w_j = 0\}.$$

(In fact  $\mathcal{M}_K^n$  is a compactification of  $\mathcal{M}_{B_n}$ ; see the proof of Corollary 8.) We let  $\pi_i: \mathbf{P}_K^n \times \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  denote the projection to the  $i$ -th factor, for  $i = 1, 2$ . The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of  $\mathbf{P}_K^n$  ( $K = \mathbf{R}$  or  $\mathbf{C}$ ) mapping  $\mathcal{M}_K^n$  into itself must be projective-linear, or possibly anti-projective-linear (if  $K = \mathbf{C}$ ):

**THEOREM 6.** *Let  $(a^1, a^2) \in \mathcal{M}_K^n$ , where  $K = \mathbf{R}$  or  $\mathbf{C}$ ,  $n \geq 2$ . Let  $U_1, U_2$  be open sets in  $\mathbf{P}_K^n$  containing  $a^1, a^2$  respectively, and let  $V_i$  be the connected component of  $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$  containing  $a_i$ , for  $i = 1, 2$ . If  $f_i: U_i \rightarrow \mathbf{P}_K^n$  ( $i = 1, 2$ ) are continuous injective maps such that*

$$(f_1 \times f_2)(\mathcal{M}_K^n \cap U_1 \times U_2) \subset \mathcal{M}_K^n,$$

*then there exists  $A \in \text{PGL}(n+1, K)$  such that*

- (i)  $f_1 = A$  on  $V_1$  and  $f_2 = {}^t A^{-1}$  on  $V_2$ , if  $K = \mathbf{R}$ ,
- (ii) either (i) holds or  $\bar{f}_1 = A$  on  $V_1$  and  $\bar{f}_2 = {}^t A^{-1}$  on  $V_2$ , if  $K = \mathbf{C}$ .

**REMARK.** If the sets  $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$  are connected, then  $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$  and we have  $\mathcal{M}_K^n \cap U_1 \times U_2 = \mathcal{M}_K^n \cap V_1 \times V_2$ . In fact, if we assume that only one of the projections  $\pi_1(\mathcal{M}_K^n \cap U_1 \times U_2)$  is connected, then by the uniqueness of  $A$  it follows that the conclusion of Theorem 6 holds with  $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ , for  $i = 1, 2$ .

*Proof of Theorem 6.* For a point  $w \in \mathbf{P}_K^n$  we write

$$w^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0\},$$

where  $z \cdot w = \sum_{j=0}^n z_j w_j$ . For a subset  $S \subset \mathbf{P}_K^n$  we also write

$$S^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0 \ \forall w \in S\}.$$

We consider the collection of lines

$$\mathcal{L}_0 = \{L \in \mathcal{L}(V_1) : L^\perp \cap U_2 \neq \emptyset\},$$

which is open in  $\mathcal{L}(V_1)$ . If  $z$  is an arbitrary point of  $V_1$ , then by hypothesis we can choose  $w \in U_2$  such that  $(z, w) \in \mathcal{M}_K^n$ . If we let  $L$  be any projective line in  $w^\perp$  containing  $z$ , then  $w \in L^\perp \cap U_2$  and hence  $L \in \mathcal{L}_0$ . Therefore  $\bigcup \mathcal{L}_0 \supset V_1$ .

Now let  $L \in \mathcal{L}_0$  be arbitrary. We claim that we can choose points  $w^1, \dots, w^{n-1} \in L^\perp \cap U_2$ , such that  $f_2(w^1), \dots, f_2(w^{n-1})$  are in general position: If  $n = 2$ , the claim is a tautology, so suppose  $n \geq 3$ . If the claim were false, then  $f_2(L^\perp \cap U_2)$  must lie in a projective linear subspace  $\mathbf{P}(E)$  of dimension  $n - 3$  (where  $E$  is a linear subspace of  $K^{n+1}$  of dimension  $n - 2$ ). But then  $f_2$  would be a continuous injection from  $(L^\perp \cap U_2)$ , which has topological dimension  $n - 2$  or  $2n - 4$  (depending on whether  $K$  equals  $\mathbf{R}$  or  $\mathbf{C}$ ), into  $\mathbf{P}(E)$ , which has topological dimension  $n - 3$  or  $2n - 6$ . This contradicts dimension theory.

Let  $w^1, \dots, w^{n-1} \in L^\perp \cap U_2$ , such that  $f_2(w^1), \dots, f_2(w^{n-1})$  are in general position, as above. By moving the points slightly if necessary, we can assume also that  $w^1, \dots, w^{n-1}$  are in general position, and hence  $L = \langle w^1, \dots, w^{n-1} \rangle^\perp$ . We note that by hypothesis,  $f_1(w^\perp \cap U_1) \subset f_2(w)^\perp$  for all  $w \in U_2$ . Therefore

$$\begin{aligned} f_1(L \cap U_1) &= \bigcap_{j=1}^{n-1} f_1(w^j \perp \cap U_1) \subset \bigcap_{j=1}^{n-1} f_2(w^j)^\perp \\ &= \langle f_2(w^1), \dots, f_2(w^{n-1}) \rangle^\perp \in \mathcal{L}_K^n(U_1). \end{aligned}$$

Let  $G$  be the group of projective-linear, and if  $K = \mathbf{C}$ , anti-projective linear, transformations of  $\mathbf{P}_K^n$  as in the proof of Theorem 3. By Theorem 3, there exists  $A \in G$  such that  $f_1 = A$  on  $V_1$ ; similarly, there exists  $B \in G$  such that  $f_2 = B$  on  $V_2$ . By replacing  $f_1 \times f_2$  with  $\bar{f}_1 \times \bar{f}_2$  if necessary, we can assume that  $A \in \text{PGL}(n+1, K)$ . We now show that  $B = {}^t A^{-1}$ : Let  $M$  be the connected component of  $\mathcal{M}_K^n \cap U_1 \times U_2$  containing  $(a^1, a^2)$ . Fix a point  $w \in \pi_2(M) \subset V_2$ , and choose  $z^1, \dots, z^n \in w^\perp \cap V_1$  in general position. Then  $(Az^j, Bw) = (f_1(z^j), f_2(w)) \in \mathcal{M}_K^n$  since  $(z^j, w) \in \mathcal{M}_K^n$ , and thus

$$0 = Az^j \cdot Bw = z^j \cdot {}^tABw,$$

for  $j = 1, \dots, n$ . Therefore  ${}^tABw \in w^{\perp\perp} = \{w\}$ . Since  $w$  is an arbitrary point of  $\pi_2(M)$  and since elements of  $G$  are uniquely determined by their values on the open set  $\pi_2(M)$ , it follows that  ${}^tAB$  is the identity  $e \in G$ , and therefore  $B = {}^tA^{-1} \in \text{PGL}(n+1, K)$ .  $\square$

COROLLARY 7 (Chern-Ji [CJ, Theorem 2]). *Suppose  $U, \hat{U}, V, \hat{V}$  are connected open sets in  $\mathbf{P}_{\mathbf{C}}^n$  such that  $\mathcal{M}_{\mathbf{C}}^n \cap U \times V \neq \emptyset$ . If  $f: U \rightarrow \hat{U}, g: V \rightarrow \hat{V}$  are biholomorphic maps such that*

$$(f \times g)(\mathcal{M}_{\mathbf{C}}^n \cap U \times V) \subset \mathcal{M}_{\mathbf{C}}^n,$$

*then  $f$  and  $g$  are restrictions of elements of  $\text{PGL}(n+1, \mathbf{C})$ .*

We conclude this paper by demonstrating how the following theorem of Poincaré and Tanaka is obtained from Corollary 7.

COROLLARY 8 (Poincaré-Tanaka Theorem) [Po], [Ta]. *Let  $B_n$  denote the unit ball in  $\mathbf{C}^n, n \geq 2$ . Suppose that  $U$  is a connected open set in  $\mathbf{C}^n$  such that  $U \cap \partial B_n \neq \emptyset$ . If  $f: U \rightarrow \mathbf{C}^n$  is a nonconstant holomorphic map such that  $f(U \cap \partial B_n) \subset \partial B_n$ , then  $f|_{U \cap B_n}$  extends to an automorphism of  $B_n$ .*

*Proof.* By an elementary argument given by H. Alexander ([A], p. 250), we can assume that the Jacobian matrix of  $f$  is nonsingular at some point  $z_0 \in U \cap \partial B_n$ . (We shall give Alexander's argument later.) By replacing  $U$  with a neighborhood of  $z_0$ , we can assume that  $f$  is injective. Let  $\tau: \mathbf{C}^n \rightarrow \mathbf{C}^n$  be the conjugation  $z \mapsto \bar{z}$ . Let  $V = \tau(U)$  and consider the holomorphic map  $g = \tau \circ f \circ \tau: V \rightarrow \mathbf{C}^n$ . We let  $\hat{U} = f(U), \hat{V} = g(V) = \tau(\hat{U})$  so that the maps  $f: U \rightarrow \hat{U}, g: V \rightarrow \hat{V}$  are biholomorphic. We let  $\psi$  denote the function on  $\mathbf{C}^n \times \mathbf{C}^n$  given by  $\psi(z, w) = \sum_{j=1}^n z_j w_j - 1$  and we consider the "Segre family"

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \psi(z, w) = 0\}.$$

Let  $S: \mathbf{C}^n \rightarrow \mathbf{C}^{2n}$  be given by  $S(z) = (z, \bar{z})$ , so that  $S^{-1}(\mathcal{M}_{B_n}) = \partial B_n$  and  $S \circ f = (f \times g) \circ S$ . Let  $\Omega = U \times V$  and  $N = S(\partial B_n) = \mathcal{M}_{B_n} \cap S(\mathbf{C}^n)$ . Then

$$(f \times g)(\Omega \cap N) = S \circ f(U \cap \partial B_n) \subset S(\partial B_n) = N \subset \mathcal{M}_{B_n}.$$

Choose a point  $z_0 \in U \cap \partial B_n$ ; then  $(z_0, \bar{z}_0) \in \Omega \cap N$ . Since  $\psi \circ (f \times g)$  vanishes on  $\Omega \cap N$  and  $N$  is a totally real submanifold of (real) dimen-

sion  $2n - 1$  in  $\mathcal{M}_{B_n}$ , it follows that  $\psi \circ (f \times g)$  vanishes on the connected component of  $\Omega \cap \mathcal{M}_{B_n}$  containing  $(z_0, \bar{z}_0)$ . After shrinking  $U$  if necessary, we can assume that  $\psi \circ (f \times g)$  vanishes on  $\Omega \cap \mathcal{M}_{B_n}$  and thus  $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$ . We consider the embedding  $\iota \times \iota: \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}_{\mathbf{C}}^n \times \mathbf{P}_{\mathbf{C}}^n$  given by  $\iota(z_1, \dots, z_n) = (\sqrt{-1}: z_1: \dots: z_n)$ , which maps  $\mathcal{M}_{B_n}$  onto a (dense open) subset of  $\mathcal{M}_{\mathbf{C}}^n$ . By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1}: \iota(U) \rightarrow \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1}: \iota(V) \rightarrow \iota(\hat{V}),$$

there exists  $A \in \text{PGL}(n+1, \mathbf{C})$  such that  $\tilde{f} = A|_{\iota(U)}$ . Thus  $f$  extends to the fractional linear map  $\iota^{-1} \circ A \circ \iota$ , which gives an automorphism of  $B_n$ .

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map  $f$  must be nonsingular at some point of  $U \cap \partial B_n$ . We begin by observing that  $f^{-1}(\partial B_n)$  is nowhere dense. Indeed, suppose on the contrary that  $f^{-1}(\partial B_n)$  contains a connected open set  $U_0$  and assume without loss of generality that  $f(z_0) = (1, 0, \dots, 0)$  for some point  $z_0 \in U_0$ . Then by the maximum principle,  $f_1 \equiv 1$  and hence  $f \equiv (1, 0, \dots, 0)$  on  $U_0$  and thus on  $U$ , contradicting the assumption that  $f$  is nonconstant. Now suppose on the contrary that the Jacobian determinant of  $f$  vanishes identically on  $U \cap \partial B_n$ . Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on  $U$ . As a consequence, the fibers of  $f$  contain no isolated points. Assume without loss of generality that  $(1, 0, \dots, 0) \in U$  and choose  $r < 1$  such that the spherical cap  $W := \{z \in B_n: \text{Re } z_1 > r\}$  is contained in  $U$ . Choose a point  $p \in W$  such that  $f(p) \notin \partial B_n$ . Let  $A$  be the connected component of  $f^{-1}(f(p)) \cap W$  that contains  $p$ ;  $A$  is an analytic subvariety of  $W$  of positive dimension. Furthermore  $\bar{A} \setminus A \subset \{z \in \mathbf{C}^n: \text{Re } z_1 = r\}$ . By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function  $\varphi: A \rightarrow \mathbf{C}$  given by  $\varphi(z) = \exp z_1$ , we conclude that  $\varphi$  is constant and thus  $\bar{A} \setminus A = \emptyset$  so that  $A$  is a compact subvariety of  $W$  of positive dimension, which is impossible.  $\square$

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