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THEOREM ON AUTOMORPHISMS OF THE BALL

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let $G = \{e, \tau\} \cdot \mathrm{PGL}(n+1, \mathbf{C})$, where $\tau: \mathbf{P}_\mathbf{C}^n \rightarrow \mathbf{P}_\mathbf{C}^n$ is given by $\tau(z) = \bar{z}$ and e is the identity map. By Lemmas 5 and 4 applied to the restrictions $f|_{U_j}$, there are transformations $A_j \in G$ such that $f|_{U_j} = A_j|_{U_j}$. Since an element of G is uniquely determined by its values on a nonempty open subset of \mathbf{P}_K^n and $(U_1 \cup \dots \cup U_j) \cap U_{j+1} \neq \emptyset$, it follows by induction that $A_j = A_1$ for all j . Hence $f = A_1|_U$. \square

3. THE POINCARÉ-TANAKA AND CHERN-JI THEOREMS

The Segre family \mathcal{M}_{B_n} mentioned in the introduction has the projective analogue

$$\mathcal{M}_K^n = \{(z, w) \in \mathbf{P}_K^n \times \mathbf{P}_K^n : \sum_{j=0}^n z_j w_j = 0\}.$$

(In fact \mathcal{M}_K^n is a compactification of \mathcal{M}_{B_n} ; see the proof of Corollary 8.) We let $\pi_i: \mathbf{P}_K^n \times \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$ denote the projection to the i -th factor, for $i = 1, 2$. The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of \mathbf{P}_K^n ($K = \mathbf{R}$ or \mathbf{C}) mapping \mathcal{M}_K^n into itself must be projective-linear, or possibly anti-projective-linear (if $K = \mathbf{C}$):

THEOREM 6. *Let $(a^1, a^2) \in \mathcal{M}_K^n$, where $K = \mathbf{R}$ or \mathbf{C} , $n \geq 2$. Let U_1, U_2 be open sets in \mathbf{P}_K^n containing a^1, a^2 respectively, and let V_i be the connected component of $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ containing a_i , for $i = 1, 2$. If $f_i: U_i \rightarrow \mathbf{P}_K^n$ ($i = 1, 2$) are continuous injective maps such that*

$$(f_1 \times f_2)(\mathcal{M}_K^n \cap U_1 \times U_2) \subset \mathcal{M}_K^n,$$

then there exists $A \in \mathrm{PGL}(n+1, K)$ such that

- (i) $f_1 = A$ on V_1 and $f_2 = {}^t A^{-1}$ on V_2 , if $K = \mathbf{R}$,
- (ii) either (i) holds or $\bar{f}_1 = A$ on V_1 and $\bar{f}_2 = {}^t A^{-1}$ on V_2 , if $K = \mathbf{C}$.

REMARK. If the sets $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ are connected, then $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ and we have $\mathcal{M}_K^n \cap U_1 \times U_2 = \mathcal{M}_K^n \cap V_1 \times V_2$. In fact, if we assume that only one of the projections $\pi_1(\mathcal{M}_K^n \cap U_1 \times U_2)$ is connected, then by the uniqueness of A it follows that the conclusion of Theorem 6 holds with $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$, for $i = 1, 2$.

Proof of Theorem 6. For a point $w \in \mathbf{P}_K^n$ we write

$$w^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0\},$$

where $z \cdot w = \sum_{j=0}^n z_j w_j$. For a subset $S \subset \mathbf{P}_K^n$ we also write

$$S^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0 \ \forall w \in S\}.$$

We consider the collection of lines

$$\mathcal{L}_0 = \{L \in \mathcal{L}(V_1) : L^\perp \cap U_2 \neq \emptyset\},$$

which is open in $\mathcal{L}(V_1)$. If z is an arbitrary point of V_1 , then by hypothesis we can choose $w \in U_2$ such that $(z, w) \in \mathcal{M}_K^n$. If we let L be any projective line in w^\perp containing z , then $w \in L^\perp \cap U_2$ and hence $L \in \mathcal{L}_0$. Therefore $\bigcup \mathcal{L}_0 \supset V_1$.

Now let $L \in \mathcal{L}_0$ be arbitrary. We claim that we can choose points $w^1, \dots, w^{n-1} \in L^\perp \cap U_2$, such that $f_2(w^1), \dots, f_2(w^{n-1})$ are in general position: If $n = 2$, the claim is a tautology, so suppose $n \geq 3$. If the claim were false, then $f_2(L^\perp \cap U_2)$ must lie in a projective linear subspace $\mathbf{P}(E)$ of dimension $n - 3$ (where E is a linear subspace of K^{n+1} of dimension $n - 2$). But then f_2 would be a continuous injection from $(L^\perp \cap U_2)$, which has topological dimension $n - 2$ or $2n - 4$ (depending on whether K equals \mathbf{R} or \mathbf{C}), into $\mathbf{P}(E)$, which has topological dimension $n - 3$ or $2n - 6$. This contradicts dimension theory.

Let $w^1, \dots, w^{n-1} \in L^\perp \cap U_2$, such that $f_2(w^1), \dots, f_2(w^{n-1})$ are in general position, as above. By moving the points slightly if necessary, we can assume also that w^1, \dots, w^{n-1} are in general position, and hence $L = \langle w^1, \dots, w^{n-1} \rangle^\perp$. We note that by hypothesis, $f_1(w^\perp \cap U_1) \subset f_2(w)^\perp$ for all $w \in U_2$. Therefore

$$\begin{aligned} f_1(L \cap U_1) &= \bigcap_{j=1}^{n-1} f_1(w^{j\perp} \cap U_1) \subset \bigcap_{j=1}^{n-1} f_2(w^j)^\perp \\ &= \langle f_2(w^1), \dots, f_2(w^{n-1}) \rangle^\perp \in \mathcal{L}_K^n(U_1). \end{aligned}$$

Let G be the group of projective-linear, and if $K = \mathbf{C}$, anti-projective linear, transformations of \mathbf{P}_K^n as in the proof of Theorem 3. By Theorem 3, there exists $A \in G$ such that $f_1 = A$ on V_1 ; similarly, there exists $B \in G$ such that $f_2 = B$ on V_2 . By replacing $f_1 \times f_2$ with $\bar{f}_1 \times \bar{f}_2$ if necessary, we can assume that $A \in \mathrm{PGL}(n+1, K)$. We now show that $B = {}^t A^{-1}$: Let M be the connected component of $\mathcal{M}_K^n \cap U_1 \times U_2$ containing (a^1, a^2) . Fix a point $w \in \pi_2(M) \subset V_2$, and choose $z^1, \dots, z^n \in w^\perp \cap V_1$ in general position. Then $(Az^j, Bw) = (f_1(z^j), f_2(w)) \in \mathcal{M}_K^n$ since $(z^j, w) \in \mathcal{M}_K^n$, and thus

$$0 = Az^j \cdot Bw = z^j \cdot {}^t ABw ,$$

for $j = 1, \dots, n$. Therefore ${}^t ABw \in w^{\perp\perp} = \{w\}$. Since w is an arbitrary point of $\pi_2(M)$ and since elements of G are uniquely determined by their values on the open set $\pi_2(M)$, it follows that ${}^t AB$ is the identity $e \in G$, and therefore $B = {}^t A^{-1} \in \mathrm{PGL}(n+1, K)$. \square

COROLLARY 7 (Chern-Ji [CJ, Theorem 2]). *Suppose U, \hat{U}, V, \hat{V} are connected open sets in \mathbf{P}_C^n such that $\mathcal{M}_C^n \cap U \times V \neq \emptyset$. If $f: U \rightarrow \hat{U}, g: V \rightarrow \hat{V}$ are biholomorphic maps such that*

$$(f \times g)(\mathcal{M}_C^n \cap U \times V) \subset \mathcal{M}_C^n ,$$

then f and g are restrictions of elements of $\mathrm{PGL}(n+1, \mathbf{C})$.

We conclude this paper by demonstrating how the following theorem of Poincaré and Tanaka is obtained from Corollary 7.

COROLLARY 8 (Poincaré-Tanaka Theorem) [Po], [Ta]. *Let B_n denote the unit ball in $\mathbf{C}^n, n \geq 2$. Suppose that U is a connected open set in \mathbf{C}^n such that $U \cap \partial B_n \neq \emptyset$. If $f: U \rightarrow \mathbf{C}^n$ is a nonconstant holomorphic map such that $f(U \cap \partial B_n) \subset \partial B_n$, then $f|_{U \cap B_n}$ extends to an automorphism of B_n .*

Proof. By an elementary argument given by H. Alexander ([A], p. 250), we can assume that the Jacobian matrix of f is nonsingular at some point $z_0 \in U \cap \partial B_n$. (We shall give Alexander's argument later.) By replacing U with a neighborhood of z_0 , we can assume that f is injective. Let $\tau: \mathbf{C}^n \rightarrow \mathbf{C}^n$ be the conjugation $z \mapsto \bar{z}$. Let $V = \tau(U)$ and consider the holomorphic map $g = \tau \circ f \circ \tau: V \rightarrow \mathbf{C}^n$. We let $\hat{U} = f(U), \hat{V} = g(V) = \tau(\hat{U})$ so that the maps $f: U \rightarrow \hat{U}, g: V \rightarrow \hat{V}$ are biholomorphic. We let ψ denote the function on $\mathbf{C}^n \times \mathbf{C}^n$ given by $\psi(z, w) = \sum_{j=1}^n z_j w_j - 1$ and we consider the “Segre family”

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \psi(z, w) = 0\} .$$

Let $S: \mathbf{C}^n \rightarrow \mathbf{C}^{2n}$ be given by $S(z) = (z, \bar{z})$, so that $S^{-1}(\mathcal{M}_{B_n}) = \partial B_n$ and $S \circ f = (f \times g) \circ S$. Let $\Omega = U \times V$ and $N = S(\partial B_n) = \mathcal{M}_{B_n} \cap S(\mathbf{C}^n)$. Then

$$(f \times g)(\Omega \cap N) = S \circ f(U \cap \partial B_n) \subset S(\partial B_n) = N \subset \mathcal{M}_{B_n} .$$

Choose a point $z_0 \in U \cap \partial B_n$; then $(z_0, \bar{z}_0) \in \Omega \cap N$. Since $\psi \circ (f \times g)$ vanishes on $\Omega \cap N$ and N is a totally real submanifold of (real) dimen-

sion $2n - 1$ in \mathcal{M}_{B_n} , it follows that $\psi \circ (f \times g)$ vanishes on the connected component of $\Omega \cap \mathcal{M}_{B_n}$ containing (z_0, \bar{z}_0) . After shrinking U if necessary, we can assume that $\psi \circ (f \times g)$ vanishes on $\Omega \cap \mathcal{M}_{B_n}$ and thus $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$. We consider the embedding $\iota \times \iota: \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}_{\mathbf{C}}^n \times \mathbf{P}_{\mathbf{C}}^n$ given by $\iota(z_1, \dots, z_n) = (\sqrt{-1}: z_1: \dots: z_n)$, which maps \mathcal{M}_{B_n} onto a (dense open) subset of $\mathcal{M}_{\mathbf{C}}^n$. By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1}: \iota(U) \rightarrow \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1}: \iota(V) \rightarrow \iota(\hat{V}),$$

there exists $A \in \mathrm{PGL}(n+1, \mathbf{C})$ such that $\tilde{f} = A|_{\iota(U)}$. Thus f extends to the fractional linear map $\iota^{-1} \circ A \circ \iota$, which gives an automorphism of B_n .

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map f must be nonsingular at some point of $U \cap \partial B_n$. We begin by observing that $f^{-1}(\partial B_n)$ is nowhere dense. Indeed, suppose on the contrary that $f^{-1}(\partial B_n)$ contains a connected open set U_0 and assume without loss of generality that $f(z_0) = (1, 0, \dots, 0)$ for some point $z_0 \in U_0$. Then by the maximum principle, $f_1 \equiv 1$ and hence $f \equiv (1, 0, \dots, 0)$ on U_0 and thus on U , contradicting the assumption that f is nonconstant. Now suppose on the contrary that the Jacobian determinant of f vanishes identically on $U \cap \partial B_n$. Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on U . As a consequence, the fibers of f contain no isolated points. Assume without loss of generality that $(1, 0, \dots, 0) \in U$ and choose $r < 1$ such that the spherical cap $W := \{z \in B_n: \mathrm{Re} z_1 > r\}$ is contained in U . Choose a point $p \in W$ such that $f(p) \notin \partial B_n$. Let A be the connected component of $f^{-1}(f(p)) \cap W$ that contains p ; A is an analytic subvariety of W of positive dimension. Furthermore $\bar{A} \setminus A \subset \{z \in \mathbf{C}^n: \mathrm{Re} z_1 = r\}$. By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function $\varphi: A \rightarrow \mathbf{C}$ given by $\varphi(z) = \exp z_1$, we conclude that φ is constant and thus $\bar{A} \setminus A = \emptyset$ so that A is a compact subvariety of W of positive dimension, which is impossible. \square

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