PROPOSITION 1.4. *Let $G$ be of type $\mathcal{F}$. If $\chi(G) \neq 0$ then $\chi_1(G; R)$ is trivial for any coefficient ring $R$.*

*Proof.* The center, $Z(G)$, is trivial, by [Got, Theorem IV.1]. Indeed, a short proof of this fact is included below as Proposition 2.4. $\square$

We end this section with the promised fourth definition of $\chi_1(X, R)$ in terms of the transfer maps of [BG], [$D_3$]. For $\gamma \in \Gamma$, consider $\Phi^\gamma : X \times S^1 \to X$ as above. This defines $\bar{\Phi}^\gamma : X \times S^1 \to X \times S^1$ by $\bar{\Phi}^\gamma(x, z) = (\Phi^\gamma(x, z), z)$ which is a fiber map with respect to the trivial fibration $X \to X \times S^1 \to S^1$. There is an associated $S$-map (the *transfer*) $\tau(\bar{\Phi}^\gamma) : \Sigma^\infty S^1_+ \to \Sigma^\infty(X \times S^1)_+$. Here, the subscript "$+$" indicates union with a disjoint basepoint and "$\Sigma^\infty$" denotes the suspension spectrum of a space. The $S$-map $\tau(\bar{F})$ induces a homomorphism in homology $\tau(\bar{\Phi}^\gamma)_* : H_*(S^1; R) \to H_*(X \times S^1; R)$.

THEOREM 1.5. *Let $R$ be a field. Then* $\chi_1(X; R) = -p_* \tau(\bar{\Phi}^\gamma)_*([S^1])$. $\square$

This is proved in §10.

## 2. DISCUSSION OF DEFINITION $A_1$

To explain where Definition $A_1$ comes from, we must review some basic facts about Hochschild homology. Then we show that the formula in Definition $A_1$ is well-defined and homotopy invariant.

Let $R$ be a commutative ground ring and let $S$ be an associative $R$-algebra with unit. If $M$ is an $S-S$ bimodule (i.e. a left and right $S$-module satisfying $(s_1 m)s_2 = s_1(m s_2)$ for all $m \in M$, and $s_1, s_2 \in S$), the *Hochschild chain complex* $\{C_*(S, M), d\}$ consists of $C_n(S, M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of $n$ copies of $S$ and

$$\begin{aligned} d(s_1 \otimes \cdots \otimes s_n \otimes m) &= s_2 \otimes \cdots \otimes s_n \otimes m s_1 \\ &\quad + \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m \\ &\quad + (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m \,. \end{aligned}$$

The tensor products are taken over $R$. The $n$-th homology of this complex is the $n$-th *Hochschild homology of $S$ with coefficient bimodule $M$*. It is denoted by $HH_n(S, M)$. If $M = S$ with the standard $S-S$ bimodule structure then we write $HH_n(S)$ for $HH_n(S, M)$.

We will be concerned mainly with $HH_1$ and $HH_0$ which are computed from

$$\begin{aligned} \cdots \to S \otimes S \otimes M &\xrightarrow{d} S \otimes M \xrightarrow{d} M \\ s_1 \otimes s_2 \otimes m &\mapsto s_2 \otimes ms_1 - s_1 s_2 \otimes m + s_1 \otimes s_2 m \\ s \otimes m &\mapsto ms - sm \end{aligned}$$

Next, we consider traces in Hochschild homology. If $A$ is a square matrix over $M$, we interpret its trace $\sum_i A_{ii}$ as an element of $M$ (i.e. as a Hochschild 0-cycle). The corresponding homology class is denoted by $T_0(A) \in HH_0(S, M)$. If $A^i, i = 1, \ldots, n$, are $q_i \times q_{i+1}$ matrices over $S$ and $B$ is a $q_{n+1} \times q_1$ matrix over $M$, we define $A^1 \otimes \cdots \otimes A^n \otimes B$ to be the $q_1 \times q_1$ matrix with entries in the $R$-module $S^{\otimes n} \otimes M$ given by

$$(A^1 \otimes \cdots \otimes A^n \otimes B)_{ij} = \sum_{k_2, \ldots, k_{n+1}} A^1_{i,k_2} \otimes A^2_{k_2,k_3} \otimes \cdots \otimes A^n_{k_n,k_{n+1}} \otimes B_{k_{n+1},j}\,.$$

The *trace* of $A^1 \otimes \cdots \otimes A^n \otimes B$, written $\operatorname{trace}(A^1 \otimes \cdots \otimes A^n \otimes B)$, is

$$\sum_{k_1, k_2, \ldots, k_{n+1}} A^1_{k_1,k_2} \otimes A^2_{k_2,k_3} \otimes \cdots \otimes A^n_{k_n,k_{n+1}} \otimes B_{k_{n+1},k_1}\,.$$

which we interpret as a Hochschild $n$-chain. Observe that the 1-chain $\operatorname{trace}(A \otimes B)$ is a cycle if and only if $\operatorname{trace}(AB) = \operatorname{trace}(BA)$, in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(S, M)$. In the application below, $S$ will be a groupring over the ground ring $R$ and $M = S$.

We will use the notation $G_1$ for the set of conjugacy classes of a group $G$. The partition of $G$ into the union of its conjugacy classes induces a direct sum decomposition of $HH_*(\mathbf{Z}G)$ as follows: each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be written in *canonical form* as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$ where we think of $g = g_1 \cdots g_n m \in G$ as "marking" the conjugacy class $C(g)$. All the generating chains occurring in the boundary $d(c)$ are easily seen to have markers in $C(g)$ when put into canonical form. For $C \in G_1$ let $C_*(\mathbf{Z}G)_C$ be the subgroup of $C_*(\mathbf{Z}G)$ generated by those generating chains whose markers lie in $C$. The decomposition $\mathbf{Z}G \cong \oplus_{C \in G_1} \mathbf{Z}C$ as a direct sum of abelian groups determines a decomposition of chain complexes $C_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} C_*(\mathbf{Z}G)_C$. There results a natural isomorphism $HH_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} HH_*(\mathbf{Z}G)_C$ where the summand $HH_*(\mathbf{Z}G)_C$ corresponds to the homology classes of Hochschild cycles marked by the elements of $C$. We call this summand the *$C$-component.* Given any $\mathbf{Z}G$-$\mathbf{Z}G$ bimodule $N$ let $\bar{N}$ be the left $\mathbf{Z}G$ module whose underlying abelian group is $N$ and whose left module structure is given by $gm = g \cdot m \cdot g^{-1}$. There is a natural isomorphism $HH_*(\mathbf{Z}G, N) \cong H_*(G, \bar{N})$

which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology; see [I, Theorem 1.d]. The decomposition $\overline{\mathbf{Z}G} \cong \oplus_{C \in G_1} \mathbf{Z}C$ is a direct sum of left $\mathbf{Z}G$ modules, inducing a direct sum decomposition $H_*(G, \overline{\mathbf{Z}G}) \cong \oplus_{C \in G_1} H_*(G, \mathbf{Z}C)$. Choosing representatives $g_C \in C$ we have an isomorphism of left $\mathbf{Z}G$ modules $\mathbf{Z}C \cong \mathbf{Z}(G/Z(g_C))$ where $Z(h) = \{g \in G \mid h = ghg^{-1}\}$ denotes the centralizer of $h \in G$. Since $H_*(G, \mathbf{Z}(G/Z(g_C)))$ is naturally isomorphic to $H_*(Z(g_C))$, we obtain a natural isomorphism $HH_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} H_*(Z(g_C))$; furthermore, $HH_*(\mathbf{Z}G)_C$ corresponds to the summand $H_*(Z(g_C))$ under this identification. In particular $HH_0(\mathbf{Z}G) \cong \mathbf{Z}G_1$, the free abelian group generated by the conjugacy classes, and $HH_1(\mathbf{Z}G) \cong \oplus_{C \in G_1} H_1(Z(g_C))$, the direct sum of the abelianizations of the centralizers. Indeed, if $g \otimes g^{-1} g_C$ is a cycle then its homology class in $HH_1(\mathbf{Z}G)$ corresponds to $\{g\} \in H_1(Z(g_C))$.

The augmentation $\varepsilon : \mathbf{Z}G \to \mathbf{Z}$ can be viewed as a morphism of $\mathbf{Z}G$-$\mathbf{Z}G$ bimodules, where $\mathbf{Z}$ is given the trivial bimodule structure, or as a morphism $\varepsilon : \overline{\mathbf{Z}G} \to \bar{\mathbf{Z}}$ of left $\mathbf{Z}G$-modules. Then there is an induced chain map $C_*(\mathbf{Z}G, \mathbf{Z}G) \overset{\varepsilon}{\to} C_*(\mathbf{Z}G, \mathbf{Z})$ and a commutative diagram:

$$\begin{array}{ccc} HH_*(\mathbf{Z}G, \mathbf{Z}G) & \overset{\varepsilon}{\to} & HH_*(\mathbf{Z}G, \mathbf{Z}) \\ \mu \downarrow & & \mu \downarrow \\ H_*(G, \overline{\mathbf{Z}G}) & \overset{\varepsilon}{\to} & H_*(G, \bar{\mathbf{Z}}) \end{array}$$

where the vertical arrows are isomorphisms.

Recall the abelianization homomorphism $A : \mathbf{Z}G \to G_{\mathrm{ab}} = H_1(X) = H_1(G)$ used in Definition $\mathrm{A}_1$.

PROPOSITION 2.1. *If* $\sum_i c_i \otimes n_i \in C_1(\mathbf{Z}G, \mathbf{Z})$ *is a Hochschild* 1-*cycle representing* $z \in HH_1(\mathbf{Z}G, \mathbf{Z})$, *where* $c_i \in \mathbf{Z}G$ *and* $n_i \in \mathbf{Z}$, *then* $\mu(z) = \sum_i A(c_i n_i) \in H_1(G)$.

*Proof.* This follows from the fact that $d : \mathbf{Z}G \otimes \mathbf{Z}G \otimes \mathbf{Z} \to \mathbf{Z}G \otimes \mathbf{Z}$ becomes $g_1 \otimes g_2 \otimes 1 \mapsto (g_2 - g_1 g_2 + g_1) \otimes 1$. One easily shows that the map $g \otimes 1 \mapsto A(g)$ induces $\mu$. $\square$

With notation as in §1, let $\tilde{D}_k^\gamma : C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$ be the lift of $D_k^\gamma$. Write $\tilde{\partial} = \oplus_k \tilde{\partial}_k$, $\tilde{D}^\gamma = \oplus(-1)^{k+1} \tilde{D}_k^\gamma$ and $\tilde{I} = \oplus_k (-1)^k \mathrm{id}_k$ (viewed as matrices). The chain homotopy relation becomes $\tilde{D}^\gamma \tilde{\partial} - \tilde{\partial} \tilde{D}^\gamma = \tilde{I}(1 - \eta_\#(\gamma)^{-1})$ [Explanation: the minus sign occurs on the left because of the sign convention built into the matrix $\tilde{D}^\gamma$; the right hand side is thus because the 0-end of the homotopy $F^\gamma$ is lifted to the identity, while the 1-end is lifted to the covering translation corresponding to $\eta_\#(\gamma)$; the inversion occurs because we have $G$ acting on the right.]

PROPOSITION 2.2. *$\chi_1(X;R)(\gamma)$, as given in Definition $A_1$, is independent of the choice of the cellular homotopy $F^\gamma$ representing $\gamma$.*

*Proof.* It is enough to consider the case $R = \mathbf{Z}$. We must show that if $F_1^\gamma \simeq F_2^\gamma : X \times I \to X \operatorname{rel} X \times \{0,1\}$, with corresponding chain homotopies $D_*^{1,\gamma}$ and $D_*^{2,\gamma}$, then $A(\operatorname{trace}(\tilde\partial D^{1,\gamma})) = A(\operatorname{trace}(\tilde\partial D^{2,\gamma}))$.

There is a degree 2 chain homotopy $\tilde E_k : C_k(\tilde X) \to C_{k+2}(\tilde X)$ such that $\tilde E_{k-1}\tilde\partial_k - \tilde\partial_{k+2}\tilde E_k = \tilde D^\gamma_{1,k} - \tilde D^\gamma_{2,k}$. Write $\tilde E = \oplus_k(-1)^{k+2}\tilde E_k$ (viewed as a matrix). Then $\tilde E\tilde\partial + \tilde\partial\tilde E = \tilde D_1^\gamma - \tilde D_2^\gamma$. So $\operatorname{trace}(\tilde\partial \otimes (\tilde D_1^\gamma - \tilde D_2^\gamma)) = d\operatorname{trace}(\tilde\partial \otimes \tilde\partial \otimes \tilde E)$ is a Hochschild boundary. The desired result now follows from Proposition 2.1. □

Direct calculation yields:

$$d(\operatorname{trace}(\tilde\partial \otimes \tilde D^\gamma)) = \chi(X)(1 - \eta_\#(\gamma)^{-1}) . \tag{2.3}$$

This leads to a quick proof (translating an idea of Stallings [St]) of an important theorem of Gottlieb [Got, Theorem IV.1]:

PROPOSITION 2.4. *If $\chi(X) \neq 0$ then $\mathscr{G}(X)$ is trivial.*

*Proof.* Since $\chi(X) \neq 0$, (2.3) shows that every $(1 - \eta_\#(\gamma)^{-1})$ represents $0 \in HH_0(\mathbf{Z}G)$. This implies that $\eta_\#(\gamma) = 1$. □

PROPOSITION 2.5. *In the Hochschild complex, $C_*(\mathbf{Z}G, \mathbf{Z}G)$, $\operatorname{trace}(\tilde\partial \otimes \tilde D^\gamma)$ is a cycle.*

*Proof.* If $\chi(X) = 0$, use (2.3). If $\chi(X) \neq 0$, use (2.3) and Proposition 2.4. □

Define the *lift* of $\chi_1(\cdot\,;\mathbf{Z})$ to be the function $\tilde{\mathrm{X}}_1(X) : \Gamma \to HH_1(\mathbf{Z}G)$ which takes $\gamma$ to $T_1(\tilde\partial \otimes \tilde D^\gamma)$, the homology class of the cycle $\operatorname{trace}(\tilde\partial \otimes \tilde D^\gamma)$. The proof of Proposition 2.2 shows that this is also independent of the choice of $F^\gamma$ representing $\gamma$.

There is a left action of $Z(G)$ on $HH_*(\mathbf{Z}G)$. At the level of chains it is defined by

$$\omega \cdot (g_1 \otimes \cdots \otimes g_n \otimes m) = g_1 \otimes \cdots \otimes g_n \otimes (m\omega^{-1})$$

where $\omega \in Z(G)$. One easily checks that this action is compatible with $d$

and hence makes $HH_*(\mathbf{Z}G)$ into a left $Z(G)$-module. The summand $HH_*(\mathbf{Z}G)_C$ is taken by the left action of $\omega$ isomorphically onto the summand $HH_*(\mathbf{Z}G)_{C\omega^{-1}}$ where $C\omega^{-1}$ is the conjugacy class $\{g\omega^{-1} \mid g \in C\}$.

Since $\eta$ maps $\Gamma$ into $Z(G)$, $\eta$ defines a left action of $\Gamma$ on $C_*(\mathbf{Z}G, \mathbf{Z}G)$ and on $HH_1(\mathbf{Z}G)$. By considering lifts of homotopies, we clearly get:

PROPOSITION 2.6. *When $HH_1(\mathbf{Z}G)$ is regarded as a left $\Gamma$-module, $\tilde{X}_1(X)$ becomes a derivation; i.e. $\tilde{X}_1(X)(\gamma_1\gamma_2) = \tilde{X}_1(X)(\gamma_1) + \gamma_1 \cdot \tilde{X}_1(X)(\gamma_2)$.* □

Derivations modulo inner derivations yield one-dimensional cohomology; in particular, $\tilde{X}_1(X)$ defines a cohomology class $\tilde{\chi}_1(X) \equiv [\tilde{X}_1(X)] \in H^1(\Gamma, HH_1(\mathbf{Z}G))$.

The derivation $\tilde{X}_1(X)$ depends on the choice of lifts $\tilde{e}$ of the cells $e$ of $X$ (see § 1). However, we have:

PROPOSITION 2.7. *Up to inner derivations, $\tilde{X}_1(X)$ is independent of the choice of cell orientations and of the choice of lifts. Hence $\tilde{\chi}_1(X)$ is a well-defined cohomology class.*

*Proof.* Another choice of cell orientations and lifts to the universal cover determines a chain complex $(C'_*(\tilde{X}), \tilde{\partial}'_*)$ and a chain homotopy $\tilde{E}^\gamma_k : C'_k(\tilde{X}) \to C'_{k+1}(\tilde{X})$. By the "change of basis formula", [GN$_1$, Proposition 3.3], we have:

$$T_1(\tilde{\partial}' \otimes \tilde{E}^\gamma) - T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) = T_1(U \otimes U^{-1}(1 - \eta_\#(\gamma)^{-1}))$$

where $U$ is the change of basis matrix. Since $\gamma \mapsto T_1(U \otimes U^{-1}(1 - \eta_\#(\gamma)^{-1}))$ is clearly an inner derivation, the conclusion follows. □

We may regard Definition A$_1$ as defining a cohomology class $\chi_1(X) \in H^1(\Gamma, H_1(G))$. Clearly we have:

PROPOSITION 2.8. *Under the homomorphism induced by $\varepsilon_* : HH_1(\mathbf{Z}G) \to H_1(G)$, $\tilde{\chi}_1(X)$ is taken to $\chi_1(X)$. Thus Definition $A_1$ is independent of the choice of lifts and $\chi_1(X)$ is homomorphism.* □

Despite Propositions 2.2 and 2.8, the formula in Definition A$_1$ might appear to depend on the CW structure of $X$. However, we have:

THEOREM 2.9. *The cohomology classes* $\tilde{\chi}_1(X)$ *and* $\chi_1(X)$ *are homotopy invariants.*

Proof. Since $\varepsilon_*(\tilde{\chi}_1(X)) = \chi_1(X)$, it is sufficient to show that $\tilde{\chi}_1(X)$ is a homotopy invariant. Let $X \to Y$ be a homotopy equivalence. By making use of mapping cylinders, we may assume without loss of generality that $X \to Y$ is an inclusion of $X$ into $Y$ as a subcomplex. Choose orientations for the cells of $Y$ and oriented lifts of these cells to the universal cover, $\tilde{Y}$, of $Y$. Let $\tilde{X} = p^{-1}(X)$ where $p: \tilde{Y} \to Y$ is the covering projection. Since $X \hookrightarrow Y$ is a homotopy equivalence, $\tilde{X}$ is the universal cover of $X$. Choose the basepoint to be a vertex of $X$. Given $\gamma \in \Gamma' = \pi_1(\mathscr{E}(Y), \mathrm{id})$, the homotopy extension property allows one to find a self homotopy of the identity $F^\gamma: Y \times I \to Y$ which has the additional property that $F^\gamma(X \times I) \subset X$. Let $\tilde{D}^\gamma_*: C_*(\tilde{Y}) \to C_*(\tilde{Y})$ be the chain homotopy determined by $F^\gamma$ and let $\tilde{D}^\gamma_* |$ be the restriction of $\tilde{D}^\gamma_*$ to $C_*(\tilde{X})$. Let $C_*(\tilde{Y}, \tilde{X})$ be the relative chain complex with boundary operator denoted by $\bar{\partial}$. Then $\tilde{D}^\gamma_*$ induces a chain homotopy on this complex which we will denote by $\bar{D}^\gamma_*$. There is a commutative diagram:

$$\begin{array}{ccccc}
C_*(\tilde{X}) & \to & C_*(\tilde{Y}) & \to & C_*(\tilde{Y}, \tilde{X}) \\
{\scriptstyle \tilde{D}^\gamma_* |} \downarrow & & {\scriptstyle \tilde{D}^\gamma_*} \downarrow & & {\scriptstyle \bar{D}^\gamma_*} \downarrow \\
C_*(\tilde{Y}) & \to & C_*(\tilde{Y}) & \to & C_*(\tilde{Y}, \tilde{X}) .
\end{array}$$

By [GN$_1$, Proposition 3.5], we have that, in $HH_1(\mathbf{Z}G)$:

$$T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) - T_1(\tilde{\partial} \,|\otimes \tilde{D}^\gamma |) = T_1(\bar{\partial} \otimes \bar{D}^\gamma) .$$

Although for a given $\gamma \in \Gamma'$, $T_1(\bar{\partial}_* \otimes \bar{D}^\gamma_*)$ could, in principle, be nonzero we will show that $\gamma \mapsto T_1(\bar{\partial}_* \otimes \bar{D}^\gamma_*)$ is a coboundary. Let $\bar{C}_* = C_*(\tilde{Y}, \tilde{X})$. Since $X \hookrightarrow Y$ is a homotopy equivalence, $\bar{C}$ is a contractible chain complex. Let $H_*: \bar{C}_* \to \bar{C}_*$ be a chain contraction. Then $\bar{D}^\gamma_*$ is chain homotopic to $H_*(1 - \eta_\#(\gamma)^{-1})$ via the chain homotopy $H_*(\bar{D}^\gamma_* - H_*(1 - \eta_\#(\gamma)^{-1}))$. Using the given bases, we can represent $\bar{\partial}$ and $H$ as matrices over $\mathbf{Z}\pi_1(Y)$. Reusing symbols, we write $\bar{\partial} = \oplus_i \bar{\partial}_i$, $H = \oplus_i (-1)^{i+1} H_i$ (viewed as matrices). Then, by [GN$_1$, Lemma 3.2], $T_1(\bar{\partial} \otimes \bar{D}^\gamma) = T_1(\bar{\partial} \otimes H(1 - \eta_\#(\gamma)^{-1}))$ where $H(1 - \eta_\#(\gamma)^{-1})$ is the matrix obtained by multiplying each element of $H$ on the right by $1 - \eta_\#(\gamma)^{-1} \in \mathbf{Z}\pi_1(Y)$. Clearly, $\gamma \mapsto T_1(\bar{\partial} \otimes H(1 - \eta_\#(\gamma)^{-1}))$ is an inner derivation. It follows that the derivations $\gamma \mapsto T_1(\tilde{\partial} \otimes \tilde{D}^\gamma)$ and $\gamma \mapsto T_1(\tilde{\partial} \,|\otimes \tilde{D}^\gamma |)$ represent the same cohomology class. □

COROLLARY 2.10. *The formula in Definition* $A_1$ *is a well-defined homotopy invariant of* $X$. □