PROPOSITION 1.4. *Let G be of type $\mathscr{F}$. If $\chi(G) \neq 0$ then $\chi_1(G; R)$ is trivial for any coefficient ring R.*

Proof. The center, $Z(G)$, is trivial, by [Got, Theorem IV.1]. Indeed, a short proof of this fact is included below as Proposition 2.4. $\square$

We end this section with the promised fourth definition of $\chi_1(X, R)$ in terms of the transfer maps of [BG], [D$_3$]. For $\gamma \in \Gamma$, consider $\Phi^\gamma : X \times S^1 \to X$ as above. This defines $\bar{\Phi}^\gamma : X \times S^1 \to X \times S^1$ by $\bar{\Phi}^\gamma(x, z) = (\Phi^\gamma(x, z), z)$ which is a fiber map with respect to the trivial fibration $X \to X \times S^1 \to S^1$. There is an associated S-map (the *transfer*) $\tau(\bar{\Phi}^\gamma) : \Sigma^\infty S^1_+ \to \Sigma^\infty (X \times S^1)_+$. Here, the subscript " $+$ " indicates union with a disjoint basepoint and "Σ^∞" denotes the suspension spectrum of a space. The S-map $\tau(\bar{F})$ induces a homomorphism in homology $\tau(\bar{\Phi}^\gamma)_* : H_*(S^1; R) \to H_*(X \times S^1; R)$.

THEOREM 1.5. *Let R be a field. Then $\chi_1(X; R) = -p_* \tau(\bar{\Phi}^\gamma)_*([S^1])$.* $\square$

This is proved in § 10.

2. DISCUSSION OF DEFINITION A$_1$

To explain where Definition A$_1$ comes from, we must review some basic facts about Hochschild homology. Then we show that the formula in Definition A$_1$ is well-defined and homotopy invariant.

Let R be a commutative ground ring and let S be an associative R-algebra with unit. If M is an $S - S$ bimodule (i.e. a left and right S-module satisfying $(s_1 m)s_2 = s_1(ms_2)$ for all $m \in M$, and $s_1, s_2 \in S$), the *Hochschild chain complex* $\{C_*(S, M), d\}$ consists of $C_n(S, M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of n copies of S and

$$d(s_1 \otimes \cdots \otimes s_n \otimes m) = s_2 \otimes \cdots \otimes s_n \otimes ms_1$$
$$+ \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m$$
$$+ (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m .$$

The tensor products are taken over R. The n-th homology of this complex is the n-th *Hochschild homology of S with coefficient bimodule M*. It is denoted by $HH_n(S, M)$. If $M = S$ with the standard $S - S$ bimodule structure then we write $HH_n(S)$ for $HH_n(S, M)$.

We will be concerned mainly with HH_1 and HH_0 which are computed from

$$\cdots \to S \otimes S \otimes M \overset{d}{\to} S \otimes M \overset{d}{\to} M$$

$$s_1 \otimes s_2 \otimes m \mapsto s_2 \otimes ms_1 - s_1 s_2 \otimes m + s_1 \otimes s_2 m$$

$$s \otimes m \mapsto ms - sm$$

Next, we consider traces in Hochschild homology. If A is a square matrix over M, we interpret its trace $\sum_i A_{ii}$ as an element of M (i.e. as a Hochschild 0-cycle). The corresponding homology class is denoted by $T_0(A) \in HH_0(S, M)$. If A^i, $i = 1, \ldots, n$, are $q_i \times q_{i+1}$ matrices over S and B is a $q_{n+1} \times q_1$ matrix over M, we define $A^1 \otimes \cdots \otimes A^n \otimes B$ to be the $q_1 \times q_1$ matrix with entries in the R-module $S^{\otimes n} \otimes M$ given by

$$(A^1 \otimes \cdots \otimes A^n \otimes B)_{ij} = \sum_{k_2, \ldots, k_{n+1}} A^1_{i, k_2} \otimes A^2_{k_2, k_3} \otimes \cdots \otimes A^n_{k_n, k_{n+1}} \otimes B_{k_{n+1}, j} \, .$$

The *trace* of $A^1 \otimes \cdots \otimes A^n \otimes B$, written $\mathrm{trace}(A^1 \otimes \cdots \otimes A^n \otimes B)$, is

$$\sum_{k_1, k_2, \ldots, k_{n+1}} A^1_{k_1, k_2} \otimes A^2_{k_2, k_3} \otimes \cdots \otimes A^n_{k_n, k_{n+1}} \otimes B_{k_{n+1}, k_1} \, .$$

which we interpret as a Hochschild n-chain. Observe that the 1-chain $\mathrm{trace}(A \otimes B)$ is a cycle if and only if $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(S, M)$. In the application below, S will be a groupring over the ground ring R and $M = S$.

We will use the notation G_1 for the set of conjugacy classes of a group G. The partition of G into the union of its conjugacy classes induces a direct sum decomposition of $HH_*(\mathbf{Z}G)$ as follows: each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be written in *canonical form* as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$ where we think of $g = g_1 \cdots g_n m \in G$ as "marking" the conjugacy class $C(g)$. All the generating chains occurring in the boundary $d(c)$ are easily seen to have markers in $C(g)$ when put into canonical form. For $C \in G_1$ let $C_*(\mathbf{Z}G)_C$ be the subgroup of $C_*(\mathbf{Z}G)$ generated by those generating chains whose markers lie in C. The decomposition $\mathbf{Z}G \cong \oplus_{C \in G_1} \mathbf{Z}C$ as a direct sum of abelian groups determines a decomposition of chain complexes $C_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} C_*(\mathbf{Z}G)_C$. There results a natural isomorphism $HH_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} HH_*(\mathbf{Z}G)_C$ where the summand $HH_*(\mathbf{Z}G)_C$ corresponds to the homology classes of Hochschild cycles marked by the elements of C. We call this summand the *C-component*. Given any $\mathbf{Z}G$-$\mathbf{Z}G$ bimodule N let $\bar{N}$ be the left $\mathbf{Z}G$ module whose underlying abelian group is N and whose left module structure is given by $gm = g \cdot m \cdot g^{-1}$. There is a natural isomorphism $HH_*(\mathbf{Z}G, N) \cong H_*(G, \bar{N})$

which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology; see [I, Theorem 1.d]. The decomposition $\overline{\mathbf{Z}G} \cong \bigoplus_{C \in G_1} \mathbf{Z}C$ is a direct sum of left $\mathbf{Z}G$ modules, inducing a direct sum decomposition $H_*(G, \overline{\mathbf{Z}G}) \cong \bigoplus_{C \in G_1} H_*(G, \mathbf{Z}C)$. Choosing representatives $g_C \in C$ we have an isomorphism of left $\mathbf{Z}G$ modules $\mathbf{Z}C \cong \mathbf{Z}(G/Z(g_C))$ where $Z(h) = \{g \in G \mid h = ghg^{-1}\}$ denotes the centralizer of $h \in G$. Since $H_*(G, \mathbf{Z}(G/Z(g_C)))$ is naturally isomorphic to $H_*(Z(g_C))$, we obtain a natural isomorphism $HH_*(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_*(Z(g_C))$; furthermore, $HH_*(\mathbf{Z}G)_C$ corresponds to the summand $H_*(Z(g_C))$ under this identification. In particular $HH_0(\mathbf{Z}G) \cong \mathbf{Z}G_1$, the free abelian group generated by the conjugacy classes, and $HH_1(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_1(Z(g_C))$, the direct sum of the abelianizations of the centralizers. Indeed, if $g \otimes g^{-1}g_C$ is a cycle then its homology class in $HH_1(\mathbf{Z}G)$ corresponds to $\{g\} \in H_1(Z(g_C))$.

The augmentation $\varepsilon: \mathbf{Z}G \to \mathbf{Z}$ can be viewed as a morphism of $\mathbf{Z}G$-$\mathbf{Z}G$ bimodules, where $\mathbf{Z}$ is given the trivial bimodule structure, or as a morphism $\varepsilon: \overline{\mathbf{Z}G} \to \overline{\mathbf{Z}}$ of left $\mathbf{Z}G$-modules. Then there is an induced chain map $C_*(\mathbf{Z}G, \mathbf{Z}G) \xrightarrow{\varepsilon} C_*(\mathbf{Z}G, \mathbf{Z})$ and a commutative diagram:

$$
\begin{array}{ccc}
HH_*(\mathbf{Z}G, \mathbf{Z}G) & \xrightarrow{\varepsilon} & HH_*(\mathbf{Z}G, \mathbf{Z}) \\
\mu \downarrow & & \mu \downarrow \\
H_*(G, \overline{\mathbf{Z}G}) & \xrightarrow{\varepsilon} & H_*(G, \overline{\mathbf{Z}})
\end{array}
$$

where the vertical arrows are isomorphisms.

Recall the abelianization homomorphism $A: \mathbf{Z}G \to G_{\mathrm{ab}} = H_1(X) = H_1(G)$ used in Definition A_1.

PROPOSITION 2.1. *If* $\sum_i c_i \otimes n_i \in C_1(\mathbf{Z}G, \mathbf{Z})$ *is a Hochschild 1-cycle representing* $z \in HH_1(\mathbf{Z}G, \mathbf{Z})$, *where* $c_i \in \mathbf{Z}G$ *and* $n_i \in \mathbf{Z}$, *then* $\mu(z) = \sum_i A(c_i n_i) \in H_1(G)$.

Proof. This follows from the fact that $d: \mathbf{Z}G \otimes \mathbf{Z}G \otimes \mathbf{Z} \to \mathbf{Z}G \otimes \mathbf{Z}$ becomes $g_1 \otimes g_2 \otimes 1 \mapsto (g_2 - g_1 g_2 + g_1) \otimes 1$. One easily shows that the map $g \otimes 1 \mapsto A(g)$ induces μ. $\square$

With notation as in §1, let $\tilde{D}_k^\gamma: C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$ be the lift of D_k^γ. Write $\tilde{\partial} = \bigoplus_k \tilde{\partial}_k$, $\tilde{D}^\gamma = \bigoplus (-1)^{k+1} \tilde{D}_k^\gamma$ and $\tilde{I} = \bigoplus_k (-1)^k \mathrm{id}_k$ (viewed as matrices). The chain homotopy relation becomes $\tilde{D}^\gamma \tilde{\partial} - \tilde{\partial} \tilde{D}^\gamma = \tilde{I}(1 - \eta_\#(\gamma)^{-1})$ [Explanation: the minus sign occurs on the left because of the sign convention built into the matrix $\tilde{D}^\gamma$; the right hand side is

thus because the 0-end of the homotopy F^γ is lifted to the identity, while the 1-end is lifted to the covering translation corresponding to $\eta_\#(\gamma)$; the inversion occurs because we have G acting on the right.]

PROPOSITION 2.2. $\chi_1(X; R)(\gamma)$, *as given in Definition* A_1, *is independent of the choice of the cellular homotopy* F^γ *representing* γ.

Proof. It is enough to consider the case $R = \mathbf{Z}$. We must show that if $F_1^\gamma \simeq F_2^\gamma: X \times I \to X \operatorname{rel} X \times \{0, 1\}$, with corresponding chain homotopies $D_*^{1,\gamma}$ and $D_*^{2,\gamma}$, then $A(\operatorname{trace}(\tilde\partial D^{1,\gamma})) = A(\operatorname{trace}(\tilde\partial D^{2,\gamma}))$.

There is a degree 2 chain homotopy $\tilde E_k: C_k(\tilde X) \to C_{k+2}(\tilde X)$ such that $\tilde E_{k-1}\tilde\partial_k - \tilde\partial_{k+2}\tilde E_k = \tilde D_{1,k}^\gamma - \tilde D_{2,k}^\gamma$. Write $\tilde E = \oplus_k(-1)^{k+2}\tilde E_k$ (viewed as a matrix). Then $\tilde E\tilde\partial + \tilde\partial\tilde E = \tilde D_1^\gamma - \tilde D_2^\gamma$. So $\operatorname{trace}(\tilde\partial \otimes (\tilde D_1^\gamma - \tilde D_2^\gamma))$ $= d\operatorname{trace}(\tilde\partial \otimes \tilde\partial \otimes \tilde E)$ is a Hochschild boundary. The desired result now follows from Proposition 2.1. $\square$

Direct calculation yields:

$$(2.3) \qquad d(\operatorname{trace}(\tilde\partial \otimes \tilde D^\gamma)) = \chi(X)(1 - \eta_\#(\gamma)^{-1}) \, .$$

This leads to a quick proof (translating an idea of Stallings [St]) of an important theorem of Gottlieb [Got, Theorem IV.1]:

PROPOSITION 2.4. *If* $\chi(X) \neq 0$ *then* $\mathscr{G}(X)$ *is trivial.*

Proof. Since $\chi(X) \neq 0$, (2.3) shows that every $(1 - \eta_\#(\gamma)^{-1})$ represents $0 \in HH_0(\mathbf{Z}G)$. This implies that $\eta_\#(\gamma) = 1$. $\square$

PROPOSITION 2.5. *In the Hochschild complex,* $C_*(\mathbf{Z}G, \mathbf{Z}G)$, $\operatorname{trace}(\tilde\partial \otimes \tilde D^\gamma)$ *is a cycle.*

Proof. If $\chi(X) = 0$, use (2.3). If $\chi(X) \neq 0$, use (2.3) and Proposition 2.4. $\square$

Define the *lift* of $\chi_1(\cdot; \mathbf{Z})$ to be the function $\tilde X_1(X): \Gamma \to HH_1(\mathbf{Z}G)$ which takes γ to $T_1(\tilde\partial \otimes \tilde D^\gamma)$, the homology class of the cycle $\operatorname{trace}(\tilde\partial \otimes \tilde D^\gamma)$. The proof of Proposition 2.2 shows that this is also independent of the choice of F^γ representing γ.

There is a left action of $Z(G)$ on $HH_*(\mathbf{Z}G)$. At the level of chains it is defined by

$$\omega \cdot (g_1 \otimes \cdots \otimes g_n \otimes m) = g_1 \otimes \cdots \otimes g_n \otimes (m\omega^{-1})$$

where $\omega \in Z(G)$. One easily checks that this action is compatible with d

and hence makes $HH_*(\mathbf{Z}G)$ into a left $Z(G)$-module. The summand $HH_*(\mathbf{Z}G)_C$ is taken by the left action of ω isomorphically onto the summand $HH_*(\mathbf{Z}G)_{C\omega^{-1}}$ where $C\omega^{-1}$ is the conjugacy class $\{g\omega^{-1} \mid g \in C\}$.

Since η maps Γ into $Z(G)$, η defines a left action of Γ on $C_*(\mathbf{Z}G, \mathbf{Z}G)$ and on $HH_1(\mathbf{Z}G)$. By considering lifts of homotopies, we clearly get:

PROPOSITION 2.6. *When* $HH_1(\mathbf{Z}G)$ *is regarded as a left* Γ-*module,* $\tilde{X}_1(X)$ *becomes a derivation; i.e.* $\tilde{X}_1(X)(\gamma_1\gamma_2) = \tilde{X}_1(X)(\gamma_1) + \gamma_1 \cdot \tilde{X}_1(X)(\gamma_2).$ $\square$

Derivations modulo inner derivations yield one-dimensional cohomology; in particular, $\tilde{X}_1(X)$ defines a cohomology class $\tilde{\chi}_1(X) \equiv [\tilde{X}_1(X)] \in H^1(\Gamma, HH_1(\mathbf{Z}G))$.

The derivation $\tilde{X}_1(X)$ depends on the choice of lifts $\tilde{e}$ of the cells e of X (see §1). However, we have:

PROPOSITION 2.7. *Up to inner derivations,* $\tilde{X}_1(X)$ *is independent of the choice of cell orientations and of the choice of lifts. Hence* $\tilde{\chi}_1(X)$ *is a well-defined cohomology class.*

Proof. Another choice of cell orientations and lifts to the universal cover determines a chain complex $(C'_*(\tilde{X}), \tilde{\partial}'_*)$ and a chain homotopy $\tilde{E}^\gamma_k : C'_k(\tilde{X}) \to C'_{k+1}(\tilde{X})$. By the "change of basis formula", [GN$_1$, Proposition 3.3], we have:

$$T_1(\tilde{\partial}' \otimes \tilde{E}^\gamma) - T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) = T_1(U \otimes U^{-1}(1 - \eta_\#(\gamma)^{-1}))$$

where U is the change of basis matrix. Since $\gamma \mapsto T_1(U \otimes U^{-1}(1 - \eta_\#(\gamma)^{-1}))$ is clearly an inner derivation, the conclusion follows. $\square$

We may regard Definition A$_1$ as defining a cohomology class $\chi_1(X) \in H^1(\Gamma, H_1(G))$. Clearly we have:

PROPOSITION 2.8. *Under the homomorphism induced by* $\varepsilon_* : HH_1(\mathbf{Z}G) \to H_1(G),$ $\tilde{\chi}_1(X)$ *is taken to* $\chi_1(X)$. *Thus Definition* A_1 *is independent of the choice of lifts and* $\chi_1(X)$ *is homomorphism.* $\square$

Despite Propositions 2.2 and 2.8, the formula in Definition A$_1$ might appear to depend on the CW structure of X. However, we have:

THEOREM 2.9. *The cohomology classes $\tilde{\chi}_1(X)$ and $\chi_1(X)$ are homotopy invariants.*

Proof. Since $\varepsilon_*(\tilde{\chi}_1(X)) = \chi_1(X)$, it is sufficient to show that $\tilde{\chi}_1(X)$ is a homotopy invariant. Let $X \to Y$ be a homotopy equivalence. By making use of mapping cylinders, we may assume without loss of generality that $X \to Y$ is an inclusion of X into Y as a subcomplex. Choose orientations for the cells of Y and oriented lifts of these cells to the universal cover, $\tilde{Y}$, of Y. Let $\tilde{X} = p^{-1}(X)$ where $p: \tilde{Y} \to Y$ is the covering projection. Since $X \hookrightarrow Y$ is a homotopy equivalence, $\tilde{X}$ is the universal cover of X. Choose the basepoint to be a vertex of X. Given $\gamma \in \Gamma' = \pi_1(\mathscr{E}(Y), \mathrm{id})$, the homotopy extension property allows one to find a self homotopy of the identity $F^\gamma: Y \times I \to Y$ which has the additional property that $F^\gamma(X \times I) \subset X$. Let $\tilde{D}^\gamma_*: C_*(\tilde{Y}) \to C_*(\tilde{Y})$ be the chain homotopy determined by F^γ and let $\tilde{D}^\gamma_* |$ be the restriction of $\tilde{D}^\gamma_*$ to $C_*(\tilde{X})$. Let $C_*(\tilde{Y}, \tilde{X})$ be the relative chain complex with boundary operator denoted by $\tilde{\partial}$. Then $\tilde{D}^\gamma_*$ induces a chain homotopy on this complex which we will denote by $\bar{D}^\gamma_*$. There is a commutative diagram:

$$\begin{array}{ccccc}
C_*(\tilde{X}) & \to & C_*(\tilde{Y}) & \to & C_*(\tilde{Y}, \tilde{X}) \\
\tilde{D}^\gamma_* | \downarrow & & \tilde{D}^\gamma_* \downarrow & & \bar{D}^\gamma_* \downarrow \\
C_*(\tilde{Y}) & \to & C_*(\tilde{Y}) & \to & C_*(\tilde{Y}, \tilde{X}).
\end{array}$$

By [GN$_1$, Proposition 3.5], we have that, in $HH_1(\mathbf{Z}G)$:

$$T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) - T_1(\tilde{\partial} | \otimes \tilde{D}^\gamma |) = T_1(\bar{\partial} \otimes \bar{D}^\gamma).$$

Although for a given $\gamma \in \Gamma'$, $T_1(\bar{\partial}_* \otimes \bar{D}^\gamma_*)$ could, in principle, be nonzero we will show that $\gamma \mapsto T_1(\bar{\partial}_* \otimes \bar{D}^\gamma_*)$ is a coboundary. Let $\bar{C}_* = C_*(\tilde{Y}, \tilde{X})$. Since $X \hookrightarrow Y$ is a homotopy equivalence, $\bar{C}$ is a contractible chain complex. Let $H_*: \bar{C}_* \to \bar{C}_*$ be a chain contraction. Then $\bar{D}^\gamma_*$ is chain homotopic to $H_*(1 - \eta_\#(\gamma)^{-1})$ via the chain homotopy $H_*(\bar{D}^\gamma_* - H_*(1 - \eta_\#(\gamma)^{-1}))$. Using the given bases, we can represent $\bar{\partial}$ and H as matrices over $\mathbf{Z}\pi_1(Y)$. Reusing symbols, we write $\bar{\partial} = \oplus_i \bar{\partial}_i$, $H = \oplus_i (-1)^{i+1} H_i$ (viewed as matrices). Then, by [GN$_1$, Lemma 3.2], $T_1(\bar{\partial} \otimes \bar{D}^\gamma) = T_1(\bar{\partial} \otimes H(1 - \eta_\#(\gamma)^{-1}))$ where $H(1 - \eta_\#(\gamma)^{-1})$ is the matrix obtained by multiplying each element of H on the right by $1 - \eta_\#(\gamma)^{-1} \in \mathbf{Z}\pi_1(Y)$. Clearly, $\gamma \mapsto T_1(\bar{\partial} \otimes H(1 - \eta_\#(\gamma)^{-1}))$ is an inner derivation. It follows that the derivations $\gamma \mapsto T_1(\tilde{\partial} \otimes \tilde{D}^\gamma)$ and $\gamma \mapsto T_1(\tilde{\partial} | \otimes \tilde{D}^\gamma |)$ represent the same cohomology class. $\square$

COROLLARY 2.10. *The formula in Definition A_1 is a well-defined homotopy invariant of X.* $\square$