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PROPOSITION 1.4. *Let  $G$  be of type  $\mathcal{F}$ . If  $\chi(G) \neq 0$  then  $\chi_1(G; R)$  is trivial for any coefficient ring  $R$ .*

*Proof.* The center,  $Z(G)$ , is trivial, by [Got, Theorem IV.1]. Indeed, a short proof of this fact is included below as Proposition 2.4.  $\square$

We end this section with the promised fourth definition of  $\chi_1(X, R)$  in terms of the transfer maps of [BG], [D<sub>3</sub>]. For  $\gamma \in \Gamma$ , consider  $\Phi^\gamma: X \times S^1 \rightarrow X$  as above. This defines  $\bar{\Phi}^\gamma: X \times S^1 \rightarrow X \times S^1$  by  $\bar{\Phi}^\gamma(x, z) = (\Phi^\gamma(x, z), z)$  which is a fiber map with respect to the trivial fibration  $X \rightarrow X \times S^1 \rightarrow S^1$ . There is an associated  $S$ -map (the *transfer*)  $\tau(\bar{\Phi}^\gamma): \Sigma^\infty S^1_+ \rightarrow \Sigma^\infty (X \times S^1)_+$ . Here, the subscript “+” indicates union with a disjoint basepoint and “ $\Sigma^\infty$ ” denotes the suspension spectrum of a space. The  $S$ -map  $\tau(\bar{F})$  induces a homomorphism in homology  $\tau(\bar{\Phi}^\gamma)_*: H_*(S^1; R) \rightarrow H_*(X \times S^1; R)$ .

THEOREM 1.5. *Let  $R$  be a field. Then  $\chi_1(X; R) = -p_*\tau(\bar{\Phi}^\gamma)_*([S^1])$ .*  $\square$

This is proved in §10.

## 2. DISCUSSION OF DEFINITION A<sub>1</sub>

To explain where Definition A<sub>1</sub> comes from, we must review some basic facts about Hochschild homology. Then we show that the formula in Definition A<sub>1</sub> is well-defined and homotopy invariant.

Let  $R$  be a commutative ground ring and let  $S$  be an associative  $R$ -algebra with unit. If  $M$  is an  $S - S$  bimodule (i.e. a left and right  $S$ -module satisfying  $(s_1 m)s_2 = s_1(ms_2)$  for all  $m \in M$ , and  $s_1, s_2 \in S$ ), the *Hochschild chain complex*  $\{C_*(S, M), d\}$  consists of  $C_n(S, M) = S^{\otimes n} \otimes M$  where  $S^{\otimes n}$  is the tensor product of  $n$  copies of  $S$  and

$$\begin{aligned} d(s_1 \otimes \cdots \otimes s_n \otimes m) &= s_2 \otimes \cdots \otimes s_n \otimes ms_1 \\ &+ \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m \\ &+ (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m. \end{aligned}$$

The tensor products are taken over  $R$ . The  $n$ -th homology of this complex is the  $n$ -th *Hochschild homology of  $S$  with coefficient bimodule  $M$* . It is denoted by  $HH_n(S, M)$ . If  $M = S$  with the standard  $S - S$  bimodule structure then we write  $HH_n(S)$  for  $HH_n(S, M)$ .

We will be concerned mainly with  $HH_1$  and  $HH_0$  which are computed from

$$\begin{aligned} \cdots &\rightarrow S \otimes S \otimes M \xrightarrow{d} S \otimes M \xrightarrow{d} M \\ s_1 \otimes s_2 \otimes m &\mapsto s_2 \otimes ms_1 - s_1 s_2 \otimes m + s_1 \otimes s_2 m \\ s \otimes m &\mapsto ms - sm \end{aligned}$$

Next, we consider traces in Hochschild homology. If  $A$  is a square matrix over  $M$ , we interpret its trace  $\sum_i A_{ii}$  as an element of  $M$  (i.e. as a Hochschild 0-cycle). The corresponding homology class is denoted by  $T_0(A) \in HH_0(S, M)$ . If  $A^i, i = 1, \dots, n$ , are  $q_i \times q_{i+1}$  matrices over  $S$  and  $B$  is a  $q_{n+1} \times q_1$  matrix over  $M$ , we define  $A^1 \otimes \cdots \otimes A^n \otimes B$  to be the  $q_1 \times q_1$  matrix with entries in the  $R$ -module  $S^{\otimes n} \otimes M$  given by

$$(A^1 \otimes \cdots \otimes A^n \otimes B)_{ij} = \sum_{k_2, \dots, k_{n+1}} A_{i, k_2}^1 \otimes A_{k_2, k_3}^2 \otimes \cdots \otimes A_{k_n, k_{n+1}}^n \otimes B_{k_{n+1}, j}.$$

The *trace* of  $A^1 \otimes \cdots \otimes A^n \otimes B$ , written  $\text{trace}(A^1 \otimes \cdots \otimes A^n \otimes B)$ , is

$$\sum_{k_1, k_2, \dots, k_{n+1}} A_{k_1, k_2}^1 \otimes A_{k_2, k_3}^2 \otimes \cdots \otimes A_{k_n, k_{n+1}}^n \otimes B_{k_{n+1}, k_1}.$$

which we interpret as a Hochschild  $n$ -chain. Observe that the 1-chain  $\text{trace}(A \otimes B)$  is a cycle if and only if  $\text{trace}(AB) = \text{trace}(BA)$ , in which case we denote its homology class by  $T_1(A \otimes B) \in HH_1(S, M)$ . In the application below,  $S$  will be a groupring over the ground ring  $R$  and  $M = S$ .

We will use the notation  $G_1$  for the set of conjugacy classes of a group  $G$ . The partition of  $G$  into the union of its conjugacy classes induces a direct sum decomposition of  $HH_*(\mathbf{Z}G)$  as follows: each generating chain  $c = g_1 \otimes \cdots \otimes g_n \otimes m$  can be written in *canonical form* as  $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$  where we think of  $g = g_1 \cdots g_n m \in G$  as “marking” the conjugacy class  $C(g)$ . All the generating chains occurring in the boundary  $d(c)$  are easily seen to have markers in  $C(g)$  when put into canonical form. For  $C \in G_1$  let  $C_*(\mathbf{Z}G)_C$  be the subgroup of  $C_*(\mathbf{Z}G)$  generated by those generating chains whose markers lie in  $C$ . The decomposition  $\mathbf{Z}G \cong \bigoplus_{C \in G_1} \mathbf{Z}C$  as a direct sum of abelian groups determines a decomposition of chain complexes  $C_*(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} C_*(\mathbf{Z}G)_C$ . There results a natural isomorphism  $HH_*(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} HH_*(\mathbf{Z}G)_C$  where the summand  $HH_*(\mathbf{Z}G)_C$  corresponds to the homology classes of Hochschild cycles marked by the elements of  $C$ . We call this summand the *C-component*. Given any  $\mathbf{Z}G$ - $\mathbf{Z}G$  bimodule  $N$  let  $\bar{N}$  be the left  $\mathbf{Z}G$  module whose underlying abelian group is  $N$  and whose left module structure is given by  $gm = g \cdot m \cdot g^{-1}$ . There is a natural isomorphism  $HH_*(\mathbf{Z}G, N) \cong H_*(G, \bar{N})$

which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology; see [I, Theorem 1.d]. The decomposition  $\overline{\mathbf{Z}G} \cong \bigoplus_{C \in G_1} \mathbf{Z}C$  is a direct sum of left  $\mathbf{Z}G$  modules, inducing a direct sum decomposition  $H_*(G, \overline{\mathbf{Z}G}) \cong \bigoplus_{C \in G_1} H_*(G, \mathbf{Z}C)$ . Choosing representatives  $g_C \in C$  we have an isomorphism of left  $\mathbf{Z}G$  modules  $\mathbf{Z}C \cong \mathbf{Z}(G/Z(g_C))$  where  $Z(h) = \{g \in G \mid h = ghg^{-1}\}$  denotes the centralizer of  $h \in G$ . Since  $H_*(G, \mathbf{Z}(G/Z(g_C)))$  is naturally isomorphic to  $H_*(Z(g_C))$ , we obtain a natural isomorphism  $HH_*(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_*(Z(g_C))$ ; furthermore,  $HH_*(\mathbf{Z}G)_C$  corresponds to the summand  $H_*(Z(g_C))$  under this identification. In particular  $HH_0(\mathbf{Z}G) \cong \mathbf{Z}G_1$ , the free abelian group generated by the conjugacy classes, and  $HH_1(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_1(Z(g_C))$ , the direct sum of the abelianizations of the centralizers. Indeed, if  $g \otimes g^{-1}g_C$  is a cycle then its homology class in  $HH_1(\mathbf{Z}G)$  corresponds to  $\{g\} \in H_1(Z(g_C))$ .

The augmentation  $\varepsilon: \mathbf{Z}G \rightarrow \mathbf{Z}$  can be viewed as a morphism of  $\mathbf{Z}G$ - $\mathbf{Z}G$  bimodules, where  $\mathbf{Z}$  is given the trivial bimodule structure, or as a morphism  $\varepsilon: \overline{\mathbf{Z}G} \rightarrow \bar{\mathbf{Z}}$  of left  $\mathbf{Z}G$ -modules. Then there is an induced chain map  $C_*(\mathbf{Z}G, \mathbf{Z}G) \xrightarrow{\varepsilon} C_*(\mathbf{Z}G, \mathbf{Z})$  and a commutative diagram:

$$\begin{array}{ccc} HH_*(\mathbf{Z}G, \mathbf{Z}G) & \xrightarrow{\varepsilon} & HH_*(\mathbf{Z}G, \mathbf{Z}) \\ \mu \downarrow & & \mu \downarrow \\ H_*(G, \overline{\mathbf{Z}G}) & \xrightarrow{\varepsilon} & H_*(G, \bar{\mathbf{Z}}) \end{array}$$

where the vertical arrows are isomorphisms.

Recall the abelianization homomorphism  $A: \mathbf{Z}G \rightarrow G_{\text{ab}} = H_1(X) = H_1(G)$  used in Definition A<sub>1</sub>.

**PROPOSITION 2.1.** *If  $\sum_i c_i \otimes n_i \in C_1(\mathbf{Z}G, \mathbf{Z})$  is a Hochschild 1-cycle representing  $z \in HH_1(\mathbf{Z}G, \mathbf{Z})$ , where  $c_i \in \mathbf{Z}G$  and  $n_i \in \mathbf{Z}$ , then  $\mu(z) = \sum_i A(c_i n_i) \in H_1(G)$ .*

*Proof.* This follows from the fact that  $d: \mathbf{Z}G \otimes \mathbf{Z}G \otimes \mathbf{Z} \rightarrow \mathbf{Z}G \otimes \mathbf{Z}$  becomes  $g_1 \otimes g_2 \otimes 1 \mapsto (g_2 - g_1 g_2 + g_1) \otimes 1$ . One easily shows that the map  $g \otimes 1 \mapsto A(g)$  induces  $\mu$ .  $\square$

With notation as in §1, let  $\tilde{D}_k^\gamma: C_k(\tilde{X}) \rightarrow C_{k+1}(\tilde{X})$  be the lift of  $D_k^\gamma$ . Write  $\tilde{\delta} = \bigoplus_k \tilde{\delta}_k$ ,  $\tilde{D}^\gamma = \bigoplus (-1)^{k+1} \tilde{D}_k^\gamma$  and  $\tilde{I} = \bigoplus_k (-1)^k \text{id}_k$  (viewed as matrices). The chain homotopy relation becomes  $\tilde{D}^\gamma \tilde{\delta} - \tilde{\delta} \tilde{D}^\gamma = \tilde{I}(1 - \eta_\#(\gamma)^{-1})$  [Explanation: the minus sign occurs on the left because of the sign convention built into the matrix  $\tilde{D}^\gamma$ ; the right hand side is

thus because the 0-end of the homotopy  $F^\gamma$  is lifted to the identity, while the 1-end is lifted to the covering translation corresponding to  $\eta_*(\gamma)$ ; the inversion occurs because we have  $G$  acting on the right.]

**PROPOSITION 2.2.**  $\chi_1(X; R)(\gamma)$ , as given in Definition A<sub>1</sub>, is independent of the choice of the cellular homotopy  $F^\gamma$  representing  $\gamma$ .

*Proof.* It is enough to consider the case  $R = \mathbf{Z}$ . We must show that if  $F_1^\gamma \simeq F_2^\gamma: X \times I \rightarrow X \text{ rel } X \times \{0, 1\}$ , with corresponding chain homotopies  $D_*^{1, \gamma}$  and  $D_*^{2, \gamma}$ , then  $A(\text{trace}(\tilde{\partial} D^{1, \gamma})) = A(\text{trace}(\tilde{\partial} D^{2, \gamma}))$ .

There is a degree 2 chain homotopy  $\tilde{E}_k: C_k(\tilde{X}) \rightarrow C_{k+2}(\tilde{X})$  such that  $\tilde{E}_{k-1} \tilde{\partial}_k - \tilde{\partial}_{k+2} \tilde{E}_k = \tilde{D}_{1, k}^\gamma - \tilde{D}_{2, k}^\gamma$ . Write  $\tilde{E} = \bigoplus_k (-1)^{k+2} \tilde{E}_k$  (viewed as a matrix). Then  $\tilde{E} \tilde{\partial} + \tilde{\partial} \tilde{E} = \tilde{D}_1^\gamma - \tilde{D}_2^\gamma$ . So  $\text{trace}(\tilde{\partial} \otimes (\tilde{D}_1^\gamma - \tilde{D}_2^\gamma)) = d \text{trace}(\tilde{\partial} \otimes \tilde{\partial} \otimes \tilde{E})$  is a Hochschild boundary. The desired result now follows from Proposition 2.1.  $\square$

Direct calculation yields:

$$(2.3) \quad d(\text{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)) = \chi(X)(1 - \eta_*(\gamma)^{-1}).$$

This leads to a quick proof (translating an idea of Stallings [St]) of an important theorem of Gottlieb [Got, Theorem IV.1]:

**PROPOSITION 2.4.** *If  $\chi(X) \neq 0$  then  $\mathcal{G}(X)$  is trivial.*

*Proof.* Since  $\chi(X) \neq 0$ , (2.3) shows that every  $(1 - \eta_*(\gamma)^{-1})$  represents  $0 \in HH_0(\mathbf{Z}G)$ . This implies that  $\eta_*(\gamma) = 1$ .  $\square$

**PROPOSITION 2.5.** *In the Hochschild complex,  $C_*(\mathbf{Z}G, \mathbf{Z}G)$ ,  $\text{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)$  is a cycle.*

*Proof.* If  $\chi(X) = 0$ , use (2.3). If  $\chi(X) \neq 0$ , use (2.3) and Proposition 2.4.  $\square$

Define the *lift* of  $\chi_1(\cdot; \mathbf{Z})$  to be the function  $\tilde{X}_1(X): \Gamma \rightarrow HH_1(\mathbf{Z}G)$  which takes  $\gamma$  to  $T_1(\tilde{\partial} \otimes \tilde{D}^\gamma)$ , the homology class of the cycle  $\text{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)$ . The proof of Proposition 2.2 shows that this is also independent of the choice of  $F^\gamma$  representing  $\gamma$ .

There is a left action of  $Z(G)$  on  $HH_*(\mathbf{Z}G)$ . At the level of chains it is defined by

$$\omega \cdot (g_1 \otimes \cdots \otimes g_n \otimes m) = g_1 \otimes \cdots \otimes g_n \otimes (m\omega^{-1})$$

where  $\omega \in Z(G)$ . One easily checks that this action is compatible with  $d$

and hence makes  $HH_*(\mathbf{Z}G)$  into a left  $Z(G)$ -module. The summand  $HH_*(\mathbf{Z}G)_C$  is taken by the left action of  $\omega$  isomorphically onto the summand  $HH_*(\mathbf{Z}G)_{C\omega^{-1}}$  where  $C\omega^{-1}$  is the conjugacy class  $\{g\omega^{-1} \mid g \in C\}$ .

Since  $\eta$  maps  $\Gamma$  into  $Z(G)$ ,  $\eta$  defines a left action of  $\Gamma$  on  $C_*(\mathbf{Z}G, \mathbf{Z}G)$  and on  $HH_1(\mathbf{Z}G)$ . By considering lifts of homotopies, we clearly get:

**PROPOSITION 2.6.** *When  $HH_1(\mathbf{Z}G)$  is regarded as a left  $\Gamma$ -module,  $\tilde{X}_1(X)$  becomes a derivation; i.e.  $\tilde{X}_1(X)(\gamma_1\gamma_2) = \tilde{X}_1(X)(\gamma_1) + \gamma_1 \cdot \tilde{X}_1(X)(\gamma_2)$ .  $\square$*

Derivations modulo inner derivations yield one-dimensional cohomology; in particular,  $\tilde{X}_1(X)$  defines a cohomology class  $\tilde{\chi}_1(X) \equiv [\tilde{X}_1(X)] \in H^1(\Gamma, HH_1(\mathbf{Z}G))$ .

The derivation  $\tilde{X}_1(X)$  depends on the choice of lifts  $\tilde{e}$  of the cells  $e$  of  $X$  (see §1). However, we have:

**PROPOSITION 2.7.** *Up to inner derivations,  $\tilde{X}_1(X)$  is independent of the choice of cell orientations and of the choice of lifts. Hence  $\tilde{\chi}_1(X)$  is a well-defined cohomology class.*

*Proof.* Another choice of cell orientations and lifts to the universal cover determines a chain complex  $(C'_*(\tilde{X}), \tilde{\partial}'_*)$  and a chain homotopy  $\tilde{E}^\gamma_k: C'_k(\tilde{X}) \rightarrow C'_{k+1}(\tilde{X})$ . By the “change of basis formula”, [GN<sub>1</sub>, Proposition 3.3], we have:

$$T_1(\tilde{\partial}' \otimes \tilde{E}^\gamma) - T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) = T_1(U \otimes U^{-1}(1 - \eta_*(\gamma)^{-1}))$$

where  $U$  is the change of basis matrix. Since  $\gamma \mapsto T_1(U \otimes U^{-1}(1 - \eta_*(\gamma)^{-1}))$  is clearly an inner derivation, the conclusion follows.  $\square$

We may regard Definition A<sub>1</sub> as defining a cohomology class  $\chi_1(X) \in H^1(\Gamma, H_1(G))$ . Clearly we have:

**PROPOSITION 2.8.** *Under the homomorphism induced by  $\varepsilon_*: HH_1(\mathbf{Z}G) \rightarrow H_1(G)$ ,  $\tilde{\chi}_1(X)$  is taken to  $\chi_1(X)$ . Thus Definition A<sub>1</sub> is independent of the choice of lifts and  $\chi_1(X)$  is a homomorphism.*  $\square$

Despite Propositions 2.2 and 2.8, the formula in Definition A<sub>1</sub> might appear to depend on the CW structure of  $X$ . However, we have:

THEOREM 2.9. *The cohomology classes  $\tilde{\chi}_1(X)$  and  $\chi_1(X)$  are homotopy invariants.*

Proof. Since  $\varepsilon_*(\tilde{\chi}_1(X)) = \chi_1(X)$ , it is sufficient to show that  $\tilde{\chi}_1(X)$  is a homotopy invariant. Let  $X \rightarrow Y$  be a homotopy equivalence. By making use of mapping cylinders, we may assume without loss of generality that  $X \rightarrow Y$  is an inclusion of  $X$  into  $Y$  as a subcomplex. Choose orientations for the cells of  $Y$  and oriented lifts of these cells to the universal cover,  $\tilde{Y}$ , of  $Y$ . Let  $\tilde{X} = p^{-1}(X)$  where  $p: \tilde{Y} \rightarrow Y$  is the covering projection. Since  $X \hookrightarrow Y$  is a homotopy equivalence,  $\tilde{X}$  is the universal cover of  $X$ . Choose the basepoint to be a vertex of  $X$ . Given  $\gamma \in \Gamma' = \pi_1(\mathcal{C}(Y), \text{id})$ , the homotopy extension property allows one to find a self homotopy of the identity  $F^\gamma: Y \times I \rightarrow Y$  which has the additional property that  $F^\gamma(X \times I) \subset X$ . Let  $\tilde{D}_*^\gamma: C_*(\tilde{Y}) \rightarrow C_*(\tilde{Y})$  be the chain homotopy determined by  $F^\gamma$  and let  $\tilde{D}_*^\gamma|$  be the restriction of  $\tilde{D}_*^\gamma$  to  $C_*(\tilde{X})$ . Let  $C_*(\tilde{Y}, \tilde{X})$  be the relative chain complex with boundary operator denoted by  $\bar{\partial}$ . Then  $\tilde{D}_*^\gamma$  induces a chain homotopy on this complex which we will denote by  $\bar{D}_*^\gamma$ . There is a commutative diagram:

$$\begin{array}{ccccc} C_*(\tilde{X}) & \rightarrow & C_*(\tilde{Y}) & \rightarrow & C_*(\tilde{Y}, \tilde{X}) \\ \tilde{D}_*^\gamma| \downarrow & & \tilde{D}_*^\gamma \downarrow & & \bar{D}_*^\gamma \downarrow \\ C_*(\tilde{Y}) & \rightarrow & C_*(\tilde{Y}) & \rightarrow & C_*(\tilde{Y}, \tilde{X}) . \end{array}$$

By [GN<sub>1</sub>, Proposition 3.5], we have that, in  $HH_1(\mathbf{Z}G)$ :

$$T_1(\bar{\partial} \otimes \tilde{D}^\gamma) - T_1(\bar{\partial}| \otimes \tilde{D}^\gamma|) = T_1(\bar{\partial} \otimes \bar{D}^\gamma) .$$

Although for a given  $\gamma \in \Gamma'$ ,  $T_1(\bar{\partial}_* \otimes \bar{D}_*^\gamma)$  could, in principle, be nonzero we will show that  $\gamma \mapsto T_1(\bar{\partial}_* \otimes \bar{D}_*^\gamma)$  is a coboundary. Let  $\bar{C}_* = C_*(\tilde{Y}, \tilde{X})$ . Since  $X \hookrightarrow Y$  is a homotopy equivalence,  $\bar{C}$  is a contractible chain complex. Let  $H_*: \bar{C}_* \rightarrow \bar{C}_*$  be a chain contraction. Then  $\bar{D}_*^\gamma$  is chain homotopic to  $H_*(1 - \eta_*(\gamma)^{-1})$  via the chain homotopy  $H_*(\bar{D}_*^\gamma - H_*(1 - \eta_*(\gamma)^{-1}))$ . Using the given bases, we can represent  $\bar{\partial}$  and  $H$  as matrices over  $\mathbf{Z}\pi_1(Y)$ . Reusing symbols, we write  $\bar{\partial} = \bigoplus_i \bar{\partial}_i$ ,  $H = \bigoplus_i (-1)^{i+1} H_i$  (viewed as matrices). Then, by [GN<sub>1</sub>, Lemma 3.2],  $T_1(\bar{\partial} \otimes \bar{D}^\gamma) = T_1(\bar{\partial} \otimes H(1 - \eta_*(\gamma)^{-1}))$  where  $H(1 - \eta_*(\gamma)^{-1})$  is the matrix obtained by multiplying each element of  $H$  on the right by  $1 - \eta_*(\gamma)^{-1} \in \mathbf{Z}\pi_1(Y)$ . Clearly,  $\gamma \mapsto T_1(\bar{\partial} \otimes H(1 - \eta_*(\gamma)^{-1}))$  is an inner derivation. It follows that the derivations  $\gamma \mapsto T_1(\bar{\partial} \otimes \bar{D}^\gamma)$  and  $\gamma \mapsto T_1(\bar{\partial}| \otimes \bar{D}^\gamma|)$  represent the same cohomology class.  $\square$

COROLLARY 2.10. *The formula in Definition A<sub>1</sub> is a well-defined homotopy invariant of  $X$ .*  $\square$