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SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

by Bernard SHIFFMAN [1])

1. INTRODUCTION

Let B_n denote the unit ball in $\mathbf{C}^n$. In 1907, Poincaré [Po] showed that any nonconstant holomorphic map f from a neighborhood $U \subset \mathbf{C}^2$ of a point $z_0 \in \partial B_2$ into $\mathbf{C}^2$ which maps $U \cap \partial B_2$ into ∂B_2 must be the restriction of an element of the Möbius group of automorphisms of B_2. This result was generalized to n variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the "Segre family" of ∂B_n,

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \textstyle\sum_{j=1}^n z_j w_j = 1\},$$

and showed that if $(z_0, w_0) \in \mathcal{M}_{B_n}$ and if f, g are nondegenerate holomorphic maps from neighborhoods U, V of z_0, w_0, respectively, into $\mathbf{C}^n$ such that $f \times g$ maps $\mathcal{M}_{B_n} \cap (U \times V)$ into $\mathcal{M}_{B_n}$, then both f and g are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point $(z_0, \bar{z}_0) \in \mathcal{M}_{B_n}$ and taking $g(w) = \overline{f(\bar{w})}$ (see § 3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in $\mathbf{C}^n$ are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are "synthetic", we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems

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of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g. [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous f, g (where the conclusion holds either for f, g or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family $\mathscr{L}_0$ of lines into a line. For the case of the real projective plane $\mathbf{P}^2_{\mathbf{R}}$, this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when $\mathscr{L}_0$ is a "4-web"; i.e., $\mathscr{L}_0$ consists of four pairwise transversal families of lines, each covering the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where $\mathscr{L}_0$ is an open set in the Grassmannian of projective lines in real or complex projective n-space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball B_2 that takes any linear section in ∂B_2 into a complex line must be either projective-linear or anti-projective-linear. Radó (see [Ra]) observed that a collineation on any subset of a projective plane $\mathbf{P}^2_K$ (over any field K) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke's theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where f is a holomorphic map from a domain $\Omega \subset B_n$ whose boundary contains an open subset of ∂B_n onto a similar domain. Alexander [Al] showed that if f has a C^∞ extension to $\bar{\Omega}$ that maps $\bar{\Omega} \cap \partial B_n$ into ∂B_n, then f extends to an automorphism of B_n; Rudin [Ru, Theorem 15.3.4] replaced Alexander's hypothesis by a much weaker condition that is satisfied, for example, when f has a continuous extension to $\bar{\Omega}$ mapping $\bar{\Omega} \cap \partial B_n$ into ∂B_n. (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

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