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## SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

by Bernard SHIFFMAN<sup>1)</sup>

### 1. INTRODUCTION

Let  $B_n$  denote the unit ball in  $\mathbf{C}^n$ . In 1907, Poincaré [Po] showed that any nonconstant holomorphic map  $f$  from a neighborhood  $U \subset \mathbf{C}^2$  of a point  $z_0 \in \partial B_2$  into  $\mathbf{C}^2$  which maps  $U \cap \partial B_2$  into  $\partial B_2$  must be the restriction of an element of the Möbius group of automorphisms of  $B_2$ . This result was generalized to  $n$  variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the “Segre family” of  $\partial B_n$ ,

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \sum_{j=1}^n z_j w_j = 1\},$$

and showed that if  $(z_0, w_0) \in \mathcal{M}_{B_n}$  and if  $f, g$  are nondegenerate holomorphic maps from neighborhoods  $U, V$  of  $z_0, w_0$ , respectively, into  $\mathbf{C}^n$  such that  $f \times g$  maps  $\mathcal{M}_{B_n} \cap (U \times V)$  into  $\mathcal{M}_{B_n}$ , then both  $f$  and  $g$  are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point  $(z_0, \bar{z}_0) \in \mathcal{M}_{B_n}$  and taking  $g(w) = \overline{f(\bar{w})}$  (see §3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in  $\mathbf{C}^n$  are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are “synthetic”, we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems

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of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g. [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous  $f, g$  (where the conclusion holds either for  $f, g$  or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family  $\mathcal{L}_0$  of lines into a line. For the case of the real projective plane  $\mathbf{P}_R^2$ , this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when  $\mathcal{L}_0$  is a “4-web”; i.e.,  $\mathcal{L}_0$  consists of four pairwise transversal families of lines, each covering the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where  $\mathcal{L}_0$  is an open set in the Grassmannian of projective lines in real or complex projective  $n$ -space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball  $B_2$  that takes any linear section in  $\partial B_2$  into a complex line must be either projective-linear or anti-projective-linear. Radó (see [Ra]) observed that a collineation on any subset of a projective plane  $\mathbf{P}_K^2$  (over any field  $K$ ) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke’s theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where  $f$  is a holomorphic map from a domain  $\Omega \subset B_n$  whose boundary contains an open subset of  $\partial B_n$  onto a similar domain. Alexander [Al] showed that if  $f$  has a  $C^\infty$  extension to  $\bar{\Omega}$  that maps  $\bar{\Omega} \cap \partial B_n$  into  $\partial B_n$ , then  $f$  extends to an automorphism of  $B_n$ ; Rudin [Ru, Theorem 15.3.4] replaced Alexander’s hypothesis by a much weaker condition that is satisfied, for example, when  $f$  has a continuous extension to  $\bar{\Omega}$  mapping  $\bar{\Omega} \cap \partial B_n$  into  $\partial B_n$ . (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

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