

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL
Autor: Shiffman, Bernard
Kapitel: 1. Introduction
DOI: <https://doi.org/10.5169/seals-61825>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

by Bernard SHIFFMAN¹⁾

1. INTRODUCTION

Let B_n denote the unit ball in \mathbf{C}^n . In 1907, Poincaré [Po] showed that any nonconstant holomorphic map f from a neighborhood $U \subset \mathbf{C}^2$ of a point $z_0 \in \partial B_2$ into \mathbf{C}^2 which maps $U \cap \partial B_2$ into ∂B_2 must be the restriction of an element of the Möbius group of automorphisms of B_2 . This result was generalized to n variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the “Segre family” of ∂B_n ,

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \sum_{j=1}^n z_j w_j = 1\},$$

and showed that if $(z_0, w_0) \in \mathcal{M}_{B_n}$ and if f, g are nondegenerate holomorphic maps from neighborhoods U, V of z_0, w_0 , respectively, into \mathbf{C}^n such that $f \times g$ maps $\mathcal{M}_{B_n} \cap (U \times V)$ into \mathcal{M}_{B_n} , then both f and g are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point $(z_0, \bar{z}_0) \in \mathcal{M}_{B_n}$ and taking $g(w) = \overline{f(\bar{w})}$ (see §3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in \mathbf{C}^n are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are “synthetic”, we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems

¹⁾ Research partially supported by National Science Foundation Grant No. DMS-9204037.

of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g. [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous f, g (where the conclusion holds either for f, g or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family \mathcal{L}_0 of lines into a line. For the case of the real projective plane \mathbf{P}_R^2 , this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when \mathcal{L}_0 is a “4-web”; i.e., \mathcal{L}_0 consists of four pairwise transversal families of lines, each covering the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where \mathcal{L}_0 is an open set in the Grassmannian of projective lines in real or complex projective n -space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball B_2 that takes any linear section in ∂B_2 into a complex line must be either projective-linear or anti-projective-linear. Radó (see [Ra]) observed that a collineation on any subset of a projective plane \mathbf{P}_K^2 (over any field K) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke’s theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where f is a holomorphic map from a domain $\Omega \subset B_n$ whose boundary contains an open subset of ∂B_n onto a similar domain. Alexander [Al] showed that if f has a C^∞ extension to $\bar{\Omega}$ that maps $\bar{\Omega} \cap \partial B_n$ into ∂B_n , then f extends to an automorphism of B_n ; Rudin [Ru, Theorem 15.3.4] replaced Alexander’s hypothesis by a much weaker condition that is satisfied, for example, when f has a continuous extension to $\bar{\Omega}$ mapping $\bar{\Omega} \cap \partial B_n$ into ∂B_n . (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

Acknowledgements. I would like to thank Valery Alexeev, Shiing-Shen Chern, Shanyu Ji and Sid Webster for their helpful suggestions and references to the literature.