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**Autor:** Shiffman, Bernard  
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## SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

by Bernard SHIFFMAN<sup>1)</sup>

### 1. INTRODUCTION

Let  $B_n$  denote the unit ball in  $\mathbf{C}^n$ . In 1907, Poincaré [Po] showed that any nonconstant holomorphic map  $f$  from a neighborhood  $U \subset \mathbf{C}^2$  of a point  $z_0 \in \partial B_2$  into  $\mathbf{C}^2$  which maps  $U \cap \partial B_2$  into  $\partial B_2$  must be the restriction of an element of the Möbius group of automorphisms of  $B_2$ . This result was generalized to  $n$  variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the “Segre family” of  $\partial B_n$ ,

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \sum_{j=1}^n z_j w_j = 1\},$$

and showed that if  $(z_0, w_0) \in \mathcal{M}_{B_n}$  and if  $f, g$  are nondegenerate holomorphic maps from neighborhoods  $U, V$  of  $z_0, w_0$ , respectively, into  $\mathbf{C}^n$  such that  $f \times g$  maps  $\mathcal{M}_{B_n} \cap (U \times V)$  into  $\mathcal{M}_{B_n}$ , then both  $f$  and  $g$  are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point  $(z_0, \bar{z}_0) \in \mathcal{M}_{B_n}$  and taking  $g(w) = \overline{f(\bar{w})}$  (see §3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in  $\mathbf{C}^n$  are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are “synthetic”, we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems

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of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g. [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous  $f, g$  (where the conclusion holds either for  $f, g$  or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family  $\mathcal{L}_0$  of lines into a line. For the case of the real projective plane  $\mathbf{P}_R^2$ , this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when  $\mathcal{L}_0$  is a “4-web”; i.e.,  $\mathcal{L}_0$  consists of four pairwise transversal families of lines, each covering the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where  $\mathcal{L}_0$  is an open set in the Grassmannian of projective lines in real or complex projective  $n$ -space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball  $B_2$  that takes any linear section in  $\partial B_2$  into a complex line must be either projective-linear or anti-projective-linear. Radó (see [Ra]) observed that a collineation on any subset of a projective plane  $\mathbf{P}_K^2$  (over any field  $K$ ) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke’s theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where  $f$  is a holomorphic map from a domain  $\Omega \subset B_n$  whose boundary contains an open subset of  $\partial B_n$  onto a similar domain. Alexander [Al] showed that if  $f$  has a  $C^\infty$  extension to  $\bar{\Omega}$  that maps  $\bar{\Omega} \cap \partial B_n$  into  $\partial B_n$ , then  $f$  extends to an automorphism of  $B_n$ ; Rudin [Ru, Theorem 15.3.4] replaced Alexander’s hypothesis by a much weaker condition that is satisfied, for example, when  $f$  has a continuous extension to  $\bar{\Omega}$  mapping  $\bar{\Omega} \cap \partial B_n$  into  $\partial B_n$ . (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

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## 2. THE LOCAL COLLINEATION THEOREM

In this section, we show that continuous local collineations of real or complex projective space are projective-linear or anti-projective-linear (Theorem 3). Our methods involve using Desargues' Theorem to extend to a global collineation and then applying the fundamental description of collineations over an arbitrary field (Proposition 1).

We let  $\mathcal{L}_K^n$  denote the set of projective lines in projective  $n$ -space  $\mathbf{P}_K^n$  over a field  $K$ . (We are interested here in the cases  $K = \mathbf{R}$  or  $\mathbf{C}$ .) Note that  $\mathcal{L}_K^n$  can be identified with the Grassmannian of 2-dimensional subspaces of  $K^{n+1}$ . A *collineation* on  $\mathbf{P}_K^n$  is a bijective self-map  $f: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  such that  $f(L) \in \mathcal{L}_K^n$  for all  $L \in \mathcal{L}_K^n$ . Examples of collineations on  $\mathbf{P}(K^{n+1})$  are provided by elements of the projective linear group  $\text{PGL}(n+1, K) = \text{GL}(n+1, K)/(K \setminus \{0\})$ . However, these are not the only collineations. We let the group  $\text{Gal}(K)$  of automorphisms of  $K$  (the Galois group of  $K$  over its prime field,  $\mathbf{Z}_p$  or  $\mathbf{Q}$ ) act on  $\mathbf{P}_K^n$  by

$$g(z) = (gz_0 : \dots : gz_n) \quad \text{for } g \in \text{Gal}(K), \quad z = (z_0 : \dots : z_n) \in \mathbf{P}_K^n;$$

then elements of  $\text{Gal}(K)$  also give collineations on  $\mathbf{P}_K^n$ . The following well-known result (see [Ar, Theorem 2.26]) states that these examples provide all the collineations on  $\mathbf{P}_K^n$ :

**PROPOSITION 1.** *Let  $f: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  be a collineation, where  $n \geq 2$  and  $K$  is an arbitrary field. Then there exist a unique  $A \in \text{PGL}(n+1, K)$  and a unique  $g \in \text{Gal}(K)$  such that  $f = g \circ A$ .*

We shall use of the following immediate consequence of Proposition 1:

**COROLLARY 2.** *Let  $f: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  be a collineation, where  $K = \mathbf{R}$  or  $\mathbf{C}$ ,  $n \geq 2$ . Suppose  $f$  is continuous on a nonempty open subset of  $\mathbf{P}_K^n$ . If  $K = \mathbf{R}$ , then  $f \in \text{PGL}(n+1, \mathbf{R})$ . If  $K = \mathbf{C}$ , then either  $f$  or  $\bar{f}$  is in  $\text{PGL}(n+1, \mathbf{C})$ .*

We let  $\langle a_1, \dots, a_m \rangle$  denote the projective linear subspace of  $\mathbf{P}_K^n$  determined by the points  $a_1, \dots, a_m \in \mathbf{P}_K^n$ . In particular,  $\langle a, b \rangle$  is the projective line through  $a$  and  $b$  (for  $a \neq b \in \mathbf{P}_K^n$ ). We also let  $a$  denote the one-point set  $\langle a \rangle = \{a\}$ . We now give a short proof of Proposition 1. First we need two well-known, elementary lemmas:

**LEMMA (a).** *Let  $f: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  be a collineation. If  $a_1, \dots, a_m$  are points in general position in  $\mathbf{P}_K^n$ , then  $f(a_1), \dots, f(a_m)$  are in general position and  $f(\langle a_1, \dots, a_m \rangle) = \langle f(a_1), \dots, f(a_m) \rangle$ .*



*Proof.* It suffices to consider  $m \leq n + 1$ . If  $m = 1$  the conclusion is just the definition of a collineation. So let  $2 \leq m \leq n + 1$  and assume by induction that the lemma has been verified for  $m - 1$  points. We write  $f(a) = \hat{a}$ . Since  $f(\langle a_1, \dots, a_{m-1} \rangle) = \langle \hat{a}_1, \dots, \hat{a}_{m-1} \rangle$  and  $f$  is injective, it follows that  $\hat{a}_m \notin \langle \hat{a}_1, \dots, \hat{a}_{m-1} \rangle$  and thus  $\hat{a}_1, \dots, \hat{a}_m$  are in general position. The second conclusion follows from the fact that  $\langle \hat{a}_1, \dots, \hat{a}_m \rangle$  is the union of lines  $\langle \hat{a}_m, b \rangle$ , where  $b$  runs through the points of  $\langle \hat{a}_1, \dots, \hat{a}_{m-1} \rangle$ .  $\square$

LEMMA (b). Let  $f: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  be a collineation. If there exists a line  $L \in \mathcal{L}_K^n$  such that  $f|_L: L \rightarrow f(L)$  is projective-linear, then  $f \in \text{PGL}(n + 1, K)$ .

*Proof.* Let  $\tilde{e}_j = (0, \dots, \overset{j\text{-th}}{1}, \dots, 0) \in K^{n+1}$ ,  $0 \leq j \leq n$ ,  $\tilde{\delta} = \tilde{e}_0 + \dots + \tilde{e}_n$ , and let  $e_0, \dots, e_n, \delta$  be the corresponding points in  $\mathbf{P}_K^n$ . Let  $f: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  be as in the hypothesis; we can assume without loss of generality that  $f|_{\langle e_0, e_1 \rangle}$  is projective-linear. By Lemma (a), the points  $f(e_0), \dots, f(e_n), f(\delta)$  are in general position. Choose representatives  $\widetilde{f(e_0)}, \dots, \widetilde{f(e_n)}, \widetilde{f(\delta)}$  in  $K^{n+1} \setminus \{0\}$  of  $f(e_0), \dots, f(e_n), f(\delta)$  respectively. Let  $\lambda_j \in K \setminus \{0\}$  ( $0 \leq j \leq n$ ) be given by  $\sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$ , and let  $T \in GL(n + 1, K)$  be given by  $T(\tilde{e}_j) = \lambda_j \widetilde{f(e_j)}$ . Then  $T(\tilde{\delta}) = \sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$ .

Let  $\varphi = T^{-1} \circ f$ . Thus the lemma is reduced to the following statement:  
(A<sub>n</sub>) Let  $\varphi: \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  be a collineation such that  $\varphi|_{\langle e_0, e_1 \rangle}$  is projective-linear,  $\varphi(e_j) = e_j$  ( $0 \leq j \leq n$ ), and  $\varphi(\delta) = \delta$ . Then  $\varphi$  is the identity.

We verify (A<sub>n</sub>) by induction on  $n$ . For  $n = 1$  the conclusion is immediate. So let  $n \geq 2$  and assume (A<sub>n-1</sub>). We write  $\mathbf{P}_K^{n-1} = \langle e_0, \dots, e_{n-1} \rangle$  and let  $\delta' = (1 : \dots : 1 : 0) \in \mathbf{P}_K^{n-1}$ ; thus  $\langle e_n, \delta \rangle \cap \mathbf{P}_K^{n-1} = \{\delta'\}$ . By Lemma (a),  $\varphi(\mathbf{P}_K^{n-1}) = \mathbf{P}_K^{n-1}$  and thus  $\varphi(\delta') = \delta'$ . Hence by (A<sub>n-1</sub>),  $\varphi$  is the identity on  $\mathbf{P}_K^{n-1}$ . If a line  $L \in \mathcal{L}_K^n$  contains a point  $b \notin \mathbf{P}_K^{n-1}$  such that  $\varphi(b) = b$ , then  $\varphi(L) = L$ , since  $L$  must contain another fixed point of  $\varphi$  in  $\mathbf{P}_K^{n-1}$ . Let  $a \in \langle e_0, e_n \rangle$ ,  $a \neq e_0$ , be arbitrary. Since  $\{a\} = \langle a, \delta \rangle \cap \langle e_0, e_n \rangle$  and the points  $\delta, e_n$  are fixed by  $\varphi$ , it follows that  $\varphi(\langle a, \delta \rangle) = \langle a, \delta \rangle$  and  $\varphi(\langle e_0, e_n \rangle) = \langle e_0, e_n \rangle$  and thus  $\varphi(a) = a$ . Finally, let  $x \in \mathbf{P}_K^n \setminus \langle e_0, e_n \rangle$  be arbitrary. Since  $\{x\} = \langle a, x \rangle \cap \langle e_n, x \rangle$ , where  $a$  is as above and  $\varphi$  fixes  $a, e_n$ , it follows as before that  $\varphi(x) = x$ .  $\square$

*Proof of Proposition 1.* Consider the usual embeddings  $\mathbf{P}_K^1 \subset \mathbf{P}_K^2 \subset \mathbf{P}_K^n$ . By Lemma (a),  $f(\mathbf{P}_K^2)$  is a projective 2-plane. Hence there exists a projective linear map  $T: f(\mathbf{P}_K^2) \rightarrow \mathbf{P}_K^2$  such that the map  $f' = T \circ f|_{\mathbf{P}_K^2}: \mathbf{P}_K^2 \rightarrow \mathbf{P}_K^2$  leaves the points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(0 : 0 : 1)$  and  $(1 : 1 : 1)$  fixed. Then,

for each  $a \in K$ , we can write  $f'(1:a:0) = (1:\hat{a}:0)$ , where  $\hat{a} \in K$ . We observe that the map  $a \mapsto \hat{a}$  is an element of  $\text{Gal}(K)$ . This follows from the fact that if  $a, b \in K$ , then  $a - b$  and  $a/b$  can be constructed from the following "projective straightedge" constructions:

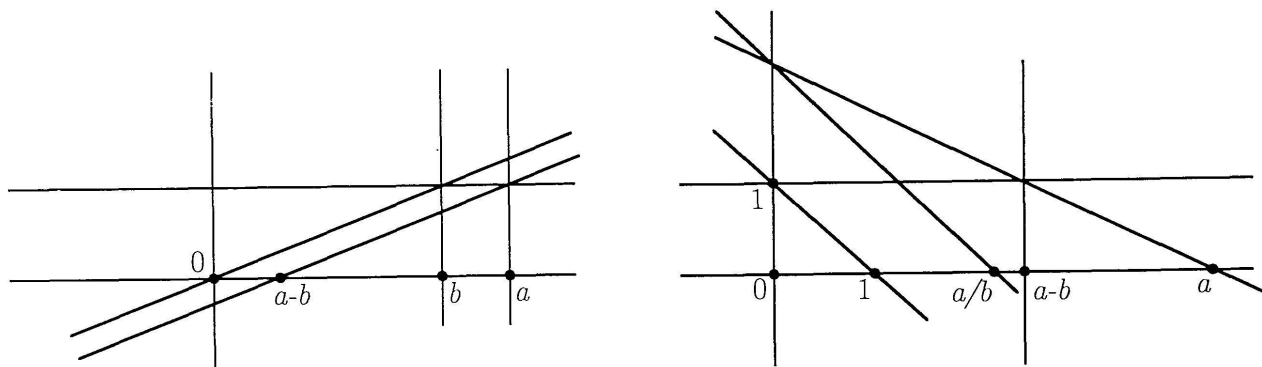


FIGURE 0

(Figure 0 shows the affine plane  $K^2 \subset \mathbf{P}_K^2$ .) Let  $g \in \text{Gal}(K)$  with  $g(a) = \hat{a}$ . Then  $f' \circ g^{-1}|_{\mathbf{P}_K^1}$  is the identity map, and it follows that the map  $f \circ g^{-1}|_{\mathbf{P}_K^1}: \mathbf{P}_K^1 \rightarrow f(\mathbf{P}_K^1)$  is projective-linear. Therefore by Lemma (b),  $f \circ g^{-1} = A' \in \text{PGL}(n+1, K)$ , and thus  $f = A' \circ g = g \circ A$ , where  $A = g^{-1}A'g \in \text{PGL}(n+1, K)$ .  $\square$

For a subset  $U \subset \mathbf{P}_K^n$ , we write

$$\mathcal{L}(U) = \{L \in \mathcal{L}_K^n : L \cap U \neq \emptyset\}.$$

We give the projective spaces  $\mathbf{P}_{\mathbf{R}}^n, \mathbf{P}_{\mathbf{C}}^n$  and the Grassmannians  $\mathcal{L}_{\mathbf{R}}^n, \mathcal{L}_{\mathbf{C}}^n$  the usual metric topologies. The main result of this section gives a condition for a local collineation to be projective-linear:

**THEOREM 3.** *Let  $U$  be a connected open set in  $\mathbf{P}_K^n (n \geq 2)$ , where  $K$  denotes either  $\mathbf{R}$  or  $\mathbf{C}$ , and let  $\mathcal{L}_0$  be an open subset of  $\mathcal{L}(U)$  such that  $\bigcup \mathcal{L}_0 \supset U$ . Suppose that  $f: U \rightarrow \mathbf{P}_K^n$  is a continuous injective map such that  $f(L \cap U)$  is contained in a projective line for all  $L \in \mathcal{L}_0$ . Then there exists  $A \in \text{PGL}(n+1, K)$  such that*

- (i)  $f = A|_U$ , if  $K = \mathbf{R}$ ,
- (ii)  $f = A|_U$  or  $\bar{f} = A|_U$ , if  $K = \mathbf{C}$ .

The case  $K = \mathbf{R}$  of Theorem 3 follows easily from Prenowitz's theorem [Pr, Theorem V], which provides a much stronger result for  $n = 2$ . (We include an elementary proof of the case  $K = \mathbf{R}$  below.)

We begin by proving the following weaker form of Theorem 3:

LEMMA 4. *Let  $U$  be an open set in  $\mathbf{P}_K^n (n \geq 2)$ , where  $K$  denotes either  $\mathbf{R}$  or  $\mathbf{C}$ , and let  $f: U \rightarrow \mathbf{P}_K^n$  be a continuous injective map. If  $f(L \cap U)$  is contained in a projective line for all  $L \in \mathcal{L}(U)$ , then the conclusion of Theorem 3 holds.*

*Proof.* Let  $f: U \rightarrow \mathbf{P}_K^n$  be as in the statement of the lemma, and let  $f(U) = \hat{U}$ . We write  $\hat{a} = f(a)$  for  $a \in U$ . Note that if three points  $a_1, a_2, a_3$  of  $U$  are not collinear, then  $\hat{a}_1, \hat{a}_2, \hat{a}_3$  are not collinear, since otherwise the sets  $f(\langle a_1, a_2 \rangle \cap U)$  and  $f(\langle a_1, a_3 \rangle \cap U)$  would both be neighborhoods of  $a_1$  in the line  $\langle \hat{a}_1, \hat{a}_2 \rangle$  and hence  $f$  would not be injective. We also observe that if  $L = \langle a, b \rangle$ , where  $a, b$  are distinct points of  $U$ , then by hypothesis,  $f(L \cap U) \subset \langle \hat{a}, \hat{b} \rangle$ , and in fact we have  $f(L \cap U) = \langle \hat{a}, \hat{b} \rangle \cap \hat{U}$ . To verify this equality, let  $\chi \in \langle \hat{a}, \hat{b} \rangle \cap \hat{U}$  be arbitrary and write  $\chi = \hat{x}$ , where  $x \in U$ . Since  $\hat{a}, \hat{b}, \hat{x}$  are collinear, it follows from the above that  $x, a, b$  are collinear and thus  $x \in L$ .

We first consider the case  $n = 2$ . Choose a connected open set  $U_0 \subset U$ . Let  $x \in \mathbf{P}_K^2$ . We want to define  $\hat{x} = \tilde{f}(x)$ . Choose  $a, b \in U_0$  such that  $a, b, x$  are not collinear. Let  $\hat{L}_a, \hat{L}_b \in \mathcal{L}(\hat{U})$  be given by  $f(\langle a, x \rangle \cap U) = \hat{L}_a \cap \hat{U}$ ,  $f(\langle b, x \rangle \cap U) = \hat{L}_b \cap \hat{U}$ . We define  $\hat{x}(a, b) \in \mathbf{P}_K^2$  by

$$\hat{L}_a \cap \hat{L}_b = \hat{x}(a, b).$$

(Note that  $\hat{L}_a \neq \hat{L}_b$  since  $\langle a, x \rangle \neq \langle b, x \rangle$  and  $f$  is injective.)

We observe that if  $a' \in \langle a, x \rangle \cap U_0$ ,  $b' \in \langle b, x \rangle \cap U_0$  with  $a' \neq a$ ,  $b' \neq b$ , then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}, \hat{b}' \rangle.$$

In particular if  $x \in U$ , then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{x} \rangle \cap \langle \hat{b}, \hat{x} \rangle = \hat{x}.$$

STEP 1.  $\hat{x}(a, b)$  is independent of the choice of  $a, b \in U_0$ .

We can assume by the above that  $x \notin U$ . Let  $a \in U_0$  and let  $b_0, b_1 \in U_0 \setminus \langle a, x \rangle$  be arbitrary. It suffices to show that  $\hat{x}(a, b_0) = \hat{x}(a, b_1)$ .

$$b'_3, b''_3, b'_2, c, b''_2$$

FIGURE 1

$$b_3^* \in \langle b'_3, x \rangle \cap \langle a'', b_2 \rangle = b_3'',$$

We note that if  $b_3 = b_2$ , then

$$b_2 = b'_3 = b''_3 = b'_2 = c = b''_2.$$

$$\begin{aligned}\hat{x}(a, b_2) &= \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_2, \hat{b}_2' \rangle = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3', \hat{b}_3'' \rangle \\ &= \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3, \hat{b}_3' \rangle = \hat{x}(a, b_3) .\end{aligned}$$

We now suppose that  $K = \mathbf{R}$ . (The proof must be modified for the case  $K = \mathbf{R}$ , since  $U_0 \setminus \langle a, x \rangle$  may not be connected.) We may assume without loss of generality that the line segment

$$C \stackrel{\text{def}}{=} \{tb_0 + (1-t)b_1 : 0 \leq t \leq 1\}$$

is contained in  $U_0$ . If  $C \cap \langle a, x \rangle = \emptyset$ , then we conclude that  $\hat{x}(a, b_0) = \hat{x}(a, b_1)$ , by the proof for the case  $K = \mathbf{C}$  above. On the other hand, if  $C \cap \langle a, x \rangle = b'$ , then

$$\hat{x}(b_0, a) = \hat{x}(b_0, b') = \hat{x}(b_0, b_1) = \hat{x}(b', b_1) = \hat{x}(a, b_1),$$

which completes Step 1 for the case  $K = \mathbf{R}$ .

We now write  $\hat{x} = \hat{x}(a, b) = \tilde{f}(x)$  for all  $x \in \mathbf{P}_K^2$ .

STEP 2.  $\tilde{f}$  is a collineation.

Let  $x, y, z$  be collinear. We must show that  $\hat{x}, \hat{y}, \hat{z}$  are collinear. Choose collinear points  $a, b, c \in U_0 \setminus \langle x, y \rangle$ . Let  $a', b', c'$  be as in Figure 2 below. We note that if  $a = b = c$ , then  $a' = b' = c' = a$ . Thus we can choose distinct collinear  $a, b, c \in U_0 \setminus \langle x, y \rangle$  such that  $a', b', c'$  are in  $U_0$ . By moving the line  $\langle a, b \rangle$  slightly if necessary, we can assume further that  $x, y, z \notin \langle a, b \rangle$ , and hence  $a', b', c'$  are distinct. By Pappas' Theorem (see for example [Co, 4.41 and Fig. 4.4a]),  $a', b', c'$  are collinear. It further follows from the above that no four of the nine labeled points in Figure 2 are collinear. By the collinearity of  $f$  on  $U$ , the points  $\hat{a}, \hat{b}, \hat{c}$  are collinear and distinct, and the same is true for  $\hat{a}', \hat{b}', \hat{c}'$ ; furthermore, no four of the points  $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}'$  are collinear. Hence  $\hat{x}, \hat{y}, \hat{z}$  are distinct, and thus  $\tilde{f}$  is injective. Applying Pappas' Theorem again (with  $a, b, c, x, y, z, a', b', c'$  replaced by  $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}', \hat{x}, \hat{y}, \hat{z}$ , respectively), we conclude that  $\hat{x}, \hat{y}, \hat{z}$  are collinear.

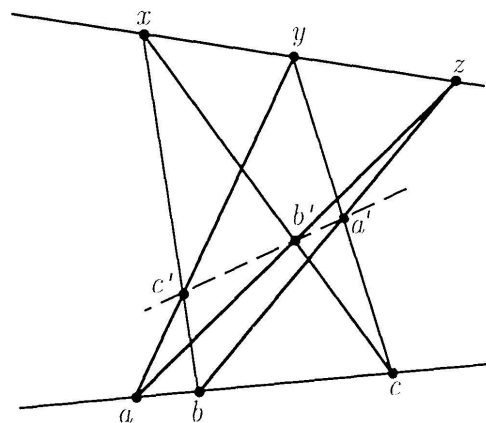


FIGURE 2

Finally, to show that  $\tilde{f}$  is surjective, let  $\chi \in \mathbf{P}_K^2$  be arbitrary. Choose points  $\alpha, \alpha', \beta, \beta' \in \hat{U}_0 = f(U_0)$  such that  $\chi = \langle \alpha, \alpha' \rangle \cap \langle \beta, \beta' \rangle$ . The points  $\alpha, \alpha', \beta, \beta'$  are the respective images of points  $a, a', b, b' \in U_0$ . If we set  $x = \langle a, a' \rangle \cap \langle b, b' \rangle$ , then  $\chi = \hat{x}$ .

Hence  $\tilde{f}$  is a collineation. The case  $n = 2$  then follows from Corollary 2.

STEP 3. *The proof for  $n > 2$ .*

Let  $n > 2$ . We easily see that  $f$  takes 2-planes in  $U$  to 2-planes in  $\hat{U}$ . Let  $L \in \mathcal{L}(U)$  be arbitrary. By applying the case  $n = 2$  to a projective 2-plane containing  $L$ , we see that  $f|_{L \cap U}: L \cap U \rightarrow \hat{L} \cap \hat{U}$  is either projective-linear or anti-projective-linear. If  $f|_{L \cap U}$  is anti-projective-linear for one  $L$ , it must be anti-projective-linear for all  $L$  (by the case  $n = 2$ ), so by replacing  $f$  with  $\bar{f}$  if necessary, we can assume that  $f|_{L \cap U}$  is projective-linear for all  $L \in \mathcal{L}(U)$ . Now fix  $a \in U$ . For  $x \in \mathbf{P}_K^n$ , define  $\hat{x} = T(x)$  where  $T: \langle a, x \rangle \rightarrow \langle \hat{a}, \hat{x} \rangle$  is the projective-linear transformation extending  $f|_{\langle a, x \rangle \cap U}$ . By applying the case  $n = 2$  to the plane determined by  $a, a', x$  (for an arbitrary point  $a' \notin \langle a, x \rangle$ ), we see that  $\hat{x}$  is independent of  $a$ . Thus we can define  $\tilde{f}(x) = \hat{x}$ . If  $x, y, z$  are collinear and  $a \notin \langle x, y \rangle$ , then the case  $n = 2$  applied to the plane determined by  $a, x, y$  implies that  $\hat{x}, \hat{y}, \hat{z}$  are collinear. The injectivity of  $\tilde{f}$  similarly follows from the case  $n = 2$ . To show surjectivity, let  $\chi \in \mathbf{P}_K^n$  be arbitrary, and choose a point  $\alpha \in \langle \hat{a}, \chi \rangle \cap \hat{U} \setminus \{\hat{a}\}$ . Then  $\alpha$  is the image of a point  $a' \in U$  and  $\tilde{f}(\langle a, a' \rangle) = \langle \hat{a}, \alpha \rangle$ . Hence  $\chi \in \langle \hat{a}, \alpha \rangle \subset \text{image } \tilde{f}$ .

Thus  $\tilde{f}$  is a collineation. The conclusion of the lemma follows as before from Corollary 2.  $\square$

DEFINITION. A subset  $U$  of  $\mathbf{P}_R^n$  or  $\mathbf{P}_C^n$  is said to be *projectively convex* if  $L \cap U$  is connected for all projective lines  $L \in \mathcal{L}(U)$ . (Note that if  $U \subset \mathbf{R}^n \subset \mathbf{P}_R^n$ , then  $U$  is projectively convex if and only if  $U$  is convex.)

We use the following lemma to complete the proof of Theorem 3:

LEMMA 5. *Let  $U$  be a projectively convex, open set in  $\mathbf{P}_K^n$ , where  $K$  denotes either  $\mathbf{R}$  or  $\mathbf{C}$ , and let  $\mathcal{L}_0$  be an open subset of  $\mathcal{L}(U)$  such that  $\bigcup \mathcal{L}_0 \supset U$ . Suppose that  $f: U \rightarrow \mathbf{P}_K^n$  is a continuous injective map such that  $f(L \cap U)$  is contained in a projective line for each  $L \in \mathcal{L}_0$ . Then  $f(L \cap U)$  is contained in a projective line for every  $L \in \mathcal{L}(U)$ .*

*Proof.* We again write  $\hat{p} = f(p)$ , for  $p \in U$ . Let  $L \in \mathcal{L}(U)$  be arbitrary, and let  $x \in L \cap U$ . Since  $L \cap U$  is connected, it suffices to show that there is a neighborhood  $V \subset U$  of  $x$  such that  $\hat{x}, \hat{y}, \hat{z}$  are collinear whenever  $y, z \in L \cap V$ . Choose a line  $L_x \in \mathcal{L}_0$  containing  $x$ . We can assume that  $L_x \neq L$ , since otherwise we are done. Choose  $w \in L_x \cap U$ ,  $w \neq x$ . Next choose a neighborhood  $V \subset U$  of  $x$  such that  $\langle y, w \rangle \in \mathcal{L}_0$  for all  $y \in V$ .

Let  $y, z \in L \cap V$ . We must show that  $\hat{x}, \hat{y}, \hat{z}$  are collinear. We can assume that  $x, y, z$  are distinct points. Choose  $v \in L \cap V$  distinct from  $x, y, z$  (see Figure 3). Since  $\langle v, w \rangle \in \mathcal{L}_0$ , we can choose  $a \in L_x \setminus \{x, w\}$  sufficiently close to  $w$  so that the line  $L_a = \langle v, a \rangle \in \mathcal{L}_0$ . Let  $b = \langle y, w \rangle \cap L_a$ ,  $c = \langle z, w \rangle \cap L_a$ . By choosing  $a$  close enough to  $w$ , we can assume further that  $a, b, c \in U$  and the six lines

$$\langle x, b \rangle, \langle x, c \rangle, \langle y, a \rangle, \langle y, c \rangle, \langle z, a \rangle, \langle z, b \rangle$$

are in  $\mathcal{L}_0$ . Let  $a', b', c'$  be as in Figure 3. Since all the points and lines of Figure 3 lie in a plane, we can use Desargues' Theorem to conclude that  $v, a', b', c'$  are collinear. Write  $L' = \langle v, c' \rangle$ ; thus  $a', b' \in L'$ . Since  $a', b', c'$  (as well as  $b, c$ ) converge to  $w$  as  $a \rightarrow w$ , by choosing  $a$  sufficiently close to  $w$  we can assume also that  $a', b', c' \in U$  and  $L' \in \mathcal{L}_0$ . Since all the labeled points in Figure 3 lie in  $U$  and all the lines in Figure 3 except  $L$  are in  $\mathcal{L}_0$ , we conclude that the  $f$ -images of the points in Figure 3 lie in the plane determined by the image lines  $\widehat{L}_a$  and  $\widehat{L}_x$ . We now apply Pappas' Theorem to the image to conclude (as in Step 2 of the proof of Lemma 4) that  $\hat{x}, \hat{y}, \hat{z}$  are collinear.  $\square$

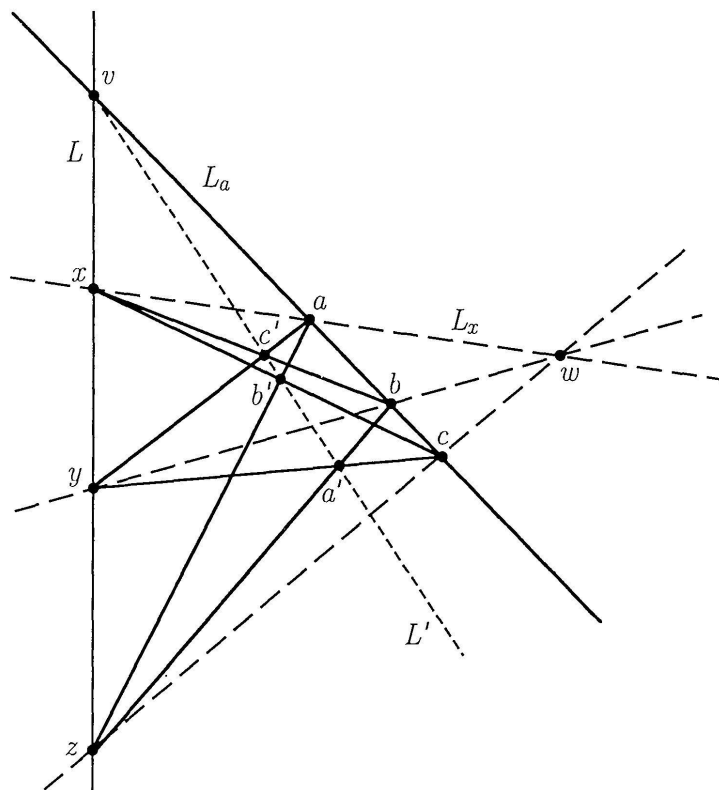


FIGURE 3

*Proof of Theorem 3.* Choose a sequence  $\{U_1, U_2, \dots\}$  of projectively convex, open subsets of  $U$  such that  $U = \bigcup_{j=1}^{\infty} U_j$  and  $U_1 \cup \dots \cup U_j$  is connected for each  $j \geq 1$ . If  $K = \mathbf{R}$ , let  $G = \text{PGL}(n+1, \mathbf{R})$ ; if  $K = \mathbf{C}$ ,

let  $G = \{e, \tau\} \cdot \text{PGL}(n+1, \mathbf{C})$ , where  $\tau: \mathbf{P}_{\mathbf{C}}^n \rightarrow \mathbf{P}_{\mathbf{C}}^n$  is given by  $\tau(z) = \bar{z}$  and  $e$  is the identity map. By Lemmas 5 and 4 applied to the restrictions  $f|_{U_j}$ , there are transformations  $A_j \in G$  such that  $f|_{U_j} = A_j|_{U_j}$ . Since an element of  $G$  is uniquely determined by its values on a nonempty open subset of  $\mathbf{P}_{\mathbf{C}}^n$  and  $(U_1 \cup \cdots \cup U_j) \cap U_{j+1} \neq \emptyset$ , it follows by induction that  $A_j = A_1$  for all  $j$ . Hence  $f = A_1|_U$ .  $\square$

### 3. THE POINCARÉ-TANAKA AND CHERN-JI THEOREMS

The Segre family  $\mathcal{M}_{B_n}$  mentioned in the introduction has the projective analogue

$$\mathcal{M}_K^n = \{(z, w) \in \mathbf{P}_K^n \times \mathbf{P}_K^n : \sum_{j=0}^n z_j w_j = 0\}.$$

(In fact  $\mathcal{M}_K^n$  is a compactification of  $\mathcal{M}_{B_n}$ ; see the proof of Corollary 8.) We let  $\pi_i: \mathbf{P}_K^n \times \mathbf{P}_K^n \rightarrow \mathbf{P}_K^n$  denote the projection to the  $i$ -th factor, for  $i = 1, 2$ . The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of  $\mathbf{P}_K^n$  ( $K = \mathbf{R}$  or  $\mathbf{C}$ ) mapping  $\mathcal{M}_K^n$  into itself must be projective-linear, or possibly anti-projective-linear (if  $K = \mathbf{C}$ ):

**THEOREM 6.** *Let  $(a^1, a^2) \in \mathcal{M}_K^n$ , where  $K = \mathbf{R}$  or  $\mathbf{C}$ ,  $n \geq 2$ . Let  $U_1, U_2$  be open sets in  $\mathbf{P}_K^n$  containing  $a^1, a^2$  respectively, and let  $V_i$  be the connected component of  $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$  containing  $a_i$ , for  $i = 1, 2$ . If  $f_i: U_i \rightarrow \mathbf{P}_K^n$  ( $i = 1, 2$ ) are continuous injective maps such that*

$$(f_1 \times f_2)(\mathcal{M}_K^n \cap U_1 \times U_2) \subset \mathcal{M}_K^n,$$

*then there exists  $A \in \text{PGL}(n+1, K)$  such that*

- (i)  $f_1 = A$  on  $V_1$  and  $f_2 = {}^t A^{-1}$  on  $V_2$ , if  $K = \mathbf{R}$ ,
- (ii) either (i) holds or  $\bar{f}_1 = A$  on  $V_1$  and  $\bar{f}_2 = {}^t A^{-1}$  on  $V_2$ , if  $K = \mathbf{C}$ .

**REMARK.** If the sets  $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$  are connected, then  $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$  and we have  $\mathcal{M}_K^n \cap U_1 \times U_2 = \mathcal{M}_K^n \cap V_1 \times V_2$ . In fact, if we assume that only one of the projections  $\pi_1(\mathcal{M}_K^n \cap U_1 \times U_2)$  is connected, then by the uniqueness of  $A$  it follows that the conclusion of Theorem 6 holds with  $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ , for  $i = 1, 2$ .



*Proof of Theorem 6.* For a point  $w \in \mathbf{P}_K^n$  we write

$$w^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0\},$$

where  $z \cdot w = \sum_{j=0}^n z_j w_j$ . For a subset  $S \subset \mathbf{P}_K^n$  we also write

$$S^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0 \ \forall w \in S\}.$$

We consider the collection of lines

$$\mathcal{L}_0 = \{L \in \mathcal{L}(V_1) : L^\perp \cap U_2 \neq \emptyset\},$$

which is open in  $\mathcal{L}(V_1)$ . If  $z$  is an arbitrary point of  $V_1$ , then by hypothesis we can choose  $w \in U_2$  such that  $(z, w) \in \mathcal{M}_K^n$ . If we let  $L$  be any projective line in  $w^\perp$  containing  $z$ , then  $w \in L^\perp \cap U_2$  and hence  $L \in \mathcal{L}_0$ . Therefore  $\bigcup \mathcal{L}_0 \supset V_1$ .

Now let  $L \in \mathcal{L}_0$  be arbitrary. We claim that we can choose points  $w^1, \dots, w^{n-1} \in L^\perp \cap U_2$ , such that  $f_2(w^1), \dots, f_2(w^{n-1})$  are in general position: If  $n = 2$ , the claim is a tautology, so suppose  $n \geq 3$ . If the claim were false, then  $f_2(L^\perp \cap U_2)$  must lie in a projective linear subspace  $\mathbf{P}(E)$  of dimension  $n - 3$  (where  $E$  is a linear subspace of  $K^{n+1}$  of dimension  $n - 2$ ). But then  $f_2$  would be a continuous injection from  $(L^\perp \cap U_2)$ , which has topological dimension  $n - 2$  or  $2n - 4$  (depending on whether  $K$  equals  $\mathbf{R}$  or  $\mathbf{C}$ ), into  $\mathbf{P}(E)$ , which has topological dimension  $n - 3$  or  $2n - 6$ . This contradicts dimension theory.

Let  $w^1, \dots, w^{n-1} \in L^\perp \cap U_2$ , such that  $f_2(w^1), \dots, f_2(w^{n-1})$  are in general position, as above. By moving the points slightly if necessary, we can assume also that  $w^1, \dots, w^{n-1}$  are in general position, and hence  $L = \langle w^1, \dots, w^{n-1} \rangle^\perp$ . We note that by hypothesis,  $f_1(w^\perp \cap U_1) \subset f_2(w)^\perp$  for all  $w \in U_2$ . Therefore

$$\begin{aligned} f_1(L \cap U_1) &= \bigcap_{j=1}^{n-1} f_1(w^j{}^\perp \cap U_1) \subset \bigcap_{j=1}^{n-1} f_2(w^j)^\perp \\ &= \langle f_2(w^1), \dots, f_2(w^{n-1}) \rangle^\perp \in \mathcal{L}_K^n(U_1). \end{aligned}$$

Let  $G$  be the group of projective-linear, and if  $K = \mathbf{C}$ , anti-projective linear, transformations of  $\mathbf{P}_K^n$  as in the proof of Theorem 3. By Theorem 3, there exists  $A \in G$  such that  $f_1 = A$  on  $V_1$ ; similarly, there exists  $B \in G$  such that  $f_2 = B$  on  $V_2$ . By replacing  $f_1 \times f_2$  with  $\bar{f}_1 \times \bar{f}_2$  if necessary, we can assume that  $A \in \text{PGL}(n+1, K)$ . We now show that  $B = {}^t A^{-1}$ : Let  $M$  be the connected component of  $\mathcal{M}_K^n \cap U_1 \times U_2$  containing  $(a^1, a^2)$ . Fix a point  $w \in \pi_2(M) \subset V_2$ , and choose  $z^1, \dots, z^n \in w^\perp \cap V_1$  in general position. Then  $(Az^j, Bw) = (f_1(z^j), f_2(w)) \in \mathcal{M}_K^n$  since  $(z^j, w) \in \mathcal{M}_K^n$ , and thus

$$0 = Az^j \cdot Bw = z^j \cdot {}^tABw ,$$

for  $j = 1, \dots, n$ . Therefore  ${}^tABw \in w^{\perp\perp} = \{w\}$ . Since  $w$  is an arbitrary point of  $\pi_2(M)$  and since elements of  $G$  are uniquely determined by their values on the open set  $\pi_2(M)$ , it follows that  ${}^tAB$  is the identity  $e \in G$ , and therefore  $B = {}^tA^{-1} \in \text{PGL}(n+1, K)$ .  $\square$

COROLLARY 7 (Chern-Ji [CJ, Theorem 2]). Suppose  $U, \hat{U}, V, \hat{V}$  are connected open sets in  $\mathbf{P}_{\mathbf{C}}^n$  such that  $\mathcal{M}_{\mathbf{C}}^n \cap U \times V \neq \emptyset$ . If  $f: U \rightarrow \hat{U}, g: V \rightarrow \hat{V}$  are biholomorphic maps such that

$$(f \times g)(\mathcal{M}_{\mathbf{C}}^n \cap U \times V) \subset \mathcal{M}_{\mathbf{C}}^n ,$$

then  $f$  and  $g$  are restrictions of elements of  $\text{PGL}(n+1, \mathbf{C})$ .

We conclude this paper by demonstrating how the following theorem of Poincaré and Tanaka is obtained from Corollary 7.

COROLLARY 8 (Poincaré-Tanaka Theorem) [Po], [Ta]. Let  $B_n$  denote the unit ball in  $\mathbf{C}^n, n \geq 2$ . Suppose that  $U$  is a connected open set in  $\mathbf{C}^n$  such that  $U \cap \partial B_n \neq \emptyset$ . If  $f: U \rightarrow \mathbf{C}^n$  is a nonconstant holomorphic map such that  $f(U \cap \partial B_n) \subset \partial B_n$ , then  $f|_{U \cap B_n}$  extends to an automorphism of  $B_n$ .

*Proof.* By an elementary argument given by H. Alexander ([A], p. 250), we can assume that the Jacobian matrix of  $f$  is nonsingular at some point  $z_0 \in U \cap \partial B_n$ . (We shall give Alexander's argument later.) By replacing  $U$  with a neighborhood of  $z_0$ , we can assume that  $f$  is injective. Let  $\tau: \mathbf{C}^n \rightarrow \mathbf{C}^n$  be the conjugation  $z \mapsto \bar{z}$ . Let  $V = \tau(U)$  and consider the holomorphic map  $g = \tau \circ f \circ \tau: V \rightarrow \mathbf{C}^n$ . We let  $\hat{U} = f(U)$ ,  $\hat{V} = g(V) = \tau(\hat{U})$  so that the maps  $f: U \rightarrow \hat{U}, g: V \rightarrow \hat{V}$  are biholomorphic. We let  $\psi$  denote the function on  $\mathbf{C}^n \times \mathbf{C}^n$  given by  $\psi(z, w) = \sum_{j=1}^n z_j \bar{w}_j - 1$  and we consider the "Segre family"

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \psi(z, w) = 0\} .$$

Let  $S: \mathbf{C}^n \rightarrow \mathbf{C}^{2n}$  be given by  $S(z) = (z, \bar{z})$ , so that  $S^{-1}(\mathcal{M}_{B_n}) = \partial B_n$  and  $S \circ f = (f \times g) \circ S$ . Let  $\Omega = U \times V$  and  $N = S(\partial B_n) = \mathcal{M}_{B_n} \cap S(\mathbf{C}^n)$ . Then

$$(f \times g)(\Omega \cap N) = S \circ f(U \cap \partial B_n) \subset S(\partial B_n) = N \subset \mathcal{M}_{B_n} .$$

Choose a point  $z_0 \in U \cap \partial B_n$ ; then  $(z_0, \bar{z}_0) \in \Omega \cap N$ . Since  $\psi \circ (f \times g)$  vanishes on  $\Omega \cap N$  and  $N$  is a totally real submanifold of (real) dimen-

sion  $2n - 1$  in  $\mathcal{M}_{B_n}$ , it follows that  $\psi \circ (f \times g)$  vanishes on the connected component of  $\Omega \cap \mathcal{M}_{B_n}$  containing  $(z_0, \bar{z}_0)$ . After shrinking  $U$  if necessary, we can assume that  $\psi \circ (f \times g)$  vanishes on  $\Omega \cap \mathcal{M}_{B_n}$  and thus  $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$ . We consider the embedding  $\iota \times \iota: \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}_{\mathbf{C}}^n \times \mathbf{P}_{\mathbf{C}}^n$  given by  $\iota(z_1, \dots, z_n) = (\sqrt{-1}: z_1: \dots: z_n)$ , which maps  $\mathcal{M}_{B_n}$  onto a (dense open) subset of  $\mathcal{M}_{\mathbf{C}}^n$ . By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1}: \iota(U) \rightarrow \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1}: \iota(V) \rightarrow \iota(\hat{V}),$$

there exists  $A \in \text{PGL}(n+1, \mathbf{C})$  such that  $\tilde{f} = A|_{\iota(U)}$ . Thus  $f$  extends to the fractional linear map  $\iota^{-1} \circ A \circ \iota$ , which gives an automorphism of  $B_n$ .

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map  $f$  must be nonsingular at some point of  $U \cap \partial B_n$ . We begin by observing that  $f^{-1}(\partial B_n)$  is nowhere dense. Indeed, suppose on the contrary that  $f^{-1}(\partial B_n)$  contains a connected open set  $U_0$  and assume without loss of generality that  $f(z_0) = (1, 0, \dots, 0)$  for some point  $z_0 \in U_0$ . Then by the maximum principle,  $f_1 \equiv 1$  and hence  $f \equiv (1, 0, \dots, 0)$  on  $U_0$  and thus on  $U$ , contradicting the assumption that  $f$  is nonconstant. Now suppose on the contrary that the Jacobian determinant of  $f$  vanishes identically on  $U \cap \partial B_n$ . Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on  $U$ . As a consequence, the fibers of  $f$  contain no isolated points. Assume without loss of generality that  $(1, 0, \dots, 0) \in U$  and choose  $r < 1$  such that the spherical cap  $W := \{z \in B_n: \text{Re } z_1 > r\}$  is contained in  $U$ . Choose a point  $p \in W$  such that  $f(p) \notin \partial B_n$ . Let  $A$  be the connected component of  $f^{-1}(f(p)) \cap W$  that contains  $p$ ;  $A$  is an analytic subvariety of  $W$  of positive dimension. Furthermore  $\bar{A} \setminus A \subset \{z \in \mathbf{C}^n: \text{Re } z_1 = r\}$ . By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function  $\varphi: A \rightarrow \mathbf{C}$  given by  $\varphi(z) = \exp z_1$ , we conclude that  $\varphi$  is constant and thus  $\bar{A} \setminus A = \emptyset$  so that  $A$  is a compact subvariety of  $W$  of positive dimension, which is impossible.  $\square$

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Bernard Shiffman

Department of Mathematics  
 Johns Hopkins University  
 Baltimore, MD 21218  
 U.S.A.

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