SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

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SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

by Bernard SHIFFMAN [1])

1. INTRODUCTION

Let B_n denote the unit ball in $\mathbf{C}^n$. In 1907, Poincaré [Po] showed that any nonconstant holomorphic map f from a neighborhood $U \subset \mathbf{C}^2$ of a point $z_0 \in \partial B_2$ into $\mathbf{C}^2$ which maps $U \cap \partial B_2$ into ∂B_2 must be the restriction of an element of the Möbius group of automorphisms of B_2. This result was generalized to n variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the "Segre family" of ∂B_n,

$$\mathscr{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \textstyle\sum_{j=1}^n z_j w_j = 1\} ,$$

and showed that if $(z_0, w_0) \in \mathscr{M}_{B_n}$ and if f, g are nondegenerate holomorphic maps from neighborhoods U, V of z_0, w_0, respectively, into $\mathbf{C}^n$ such that $f \times g$ maps $\mathscr{M}_{B_n} \cap (U \times V)$ into $\mathscr{M}_{B_n}$, then both f and g are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point $(z_0, \bar{z}_0) \in \mathscr{M}_{B_n}$ and taking $g(w) = \overline{f(\bar{w})}$ (see § 3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in $\mathbf{C}^n$ are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are "synthetic", we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems

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of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g. [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous f, g (where the conclusion holds either for f, g or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family $\mathscr{L}_0$ of lines into a line. For the case of the real projective plane $\mathbf{P}^2_{\mathbf{R}}$, this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when $\mathscr{L}_0$ is a "4-web"; i.e., $\mathscr{L}_0$ consists of four pairwise transversal families of lines, each covering the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where $\mathscr{L}_0$ is an open set in the Grassmannian of projective lines in real or complex projective n-space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball B_2 that takes any linear section in ∂B_2 into a complex line must be either projective-linear or anti-projective-linear. Radó (see [Ra]) observed that a collineation on any subset of a projective plane $\mathbf{P}^2_K$ (over any field K) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke's theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where f is a holomorphic map from a domain $\Omega \subset B_n$ whose boundary contains an open subset of ∂B_n onto a similar domain. Alexander [Al] showed that if f has a C^∞ extension to $\bar{\Omega}$ that maps $\bar{\Omega} \cap \partial B_n$ into ∂B_n, then f extends to an automorphism of B_n; Rudin [Ru, Theorem 15.3.4] replaced Alexander's hypothesis by a much weaker condition that is satisfied, for example, when f has a continuous extension to $\bar{\Omega}$ mapping $\bar{\Omega} \cap \partial B_n$ into ∂B_n. (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

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2. The local collineation theorem

In this section, we show that continuous local collineations of real or complex projective space are projective-linear or anti-projective-linear (Theorem 3). Our methods involve using Desargues' Theorem to extend to a global collineation and then applying the fundamental description of collineations over an arbitrary field (Proposition 1).

We let $\mathscr{L}^n_K$ denote the set of projective lines in projective n-space $\mathbf{P}^n_K$ over a field K. (We are interested here in the cases $K = \mathbf{R}$ or $\mathbf{C}$.) Note that $\mathscr{L}^n_K$ can be identified with the Grassmannian of 2-dimensional subspaces of K^{n+1}. A *collineation* on $\mathbf{P}^n_K$ is a bijective self-map $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ such that $f(L) \in \mathscr{L}^n_K$ for all $L \in \mathscr{L}^n_K$. Examples of collineations on $\mathbf{P}(K^{n+1})$ are provided by elements of the projective linear group $\mathrm{PGL}(n+1, K) = \mathrm{GL}(n+1, K)/(K \backslash \{0\})$. However, these are not the only collineations. We let the group $\mathrm{Gal}(K)$ of automorphisms of K (the Galois group of K over its prime field, $\mathbf{Z}_p$ or $\mathbf{Q}$) act on $\mathbf{P}^n_K$ by

$$g(z) = (gz_0 : \ldots : gz_n) \quad \text{for} \quad g \in \mathrm{Gal}(K), \quad z = (z_0 : \cdots : z_n) \in \mathbf{P}^n_K ;$$

then elements of $\mathrm{Gal}(K)$ also give collineations on $\mathbf{P}^n_K$. The following well-known result (see [Ar, Theorem 2.26]) states that these examples provide all the collineations on $\mathbf{P}^n_K$:

PROPOSITION 1. *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation, where $n \geqslant 2$ and K is an arbitrary field. Then there exist a unique $A \in \mathrm{PGL}(n+1, K)$ and a unique $g \in \mathrm{Gal}(K)$ such that $f = g \circ A$.*

We shall use of the following immediate consequence of Proposition 1:

COROLLARY 2. *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation, where $K = \mathbf{R}$ or $\mathbf{C}, n \geqslant 2$. Suppose f is continuous on a nonempty open subset of $\mathbf{P}^n_K$. If $K = \mathbf{R}$, then $f \in \mathrm{PGL}(n+1, \mathbf{R})$. If $K = \mathbf{C}$, then either f or $\bar{f}$ is in $\mathrm{PGL}(n+1, \mathbf{C})$.*

We let $\langle a_1, \ldots, a_m \rangle$ denote the projective linear subspace of $\mathbf{P}^n_K$ determined by the points $a_1, \ldots, a_m \in \mathbf{P}^n_K$. In particular, $\langle a, b \rangle$ is the projective line through a and b (for $a \neq b \in \mathbf{P}^n_K$). We also let a denote the one-point set $\langle a \rangle = \{a\}$. We now give a short proof of Proposition 1. First we need two well-known, elementary lemmas:

LEMMA (a). *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation. If $a_1, \ldots, a_m$ are points in general position in $\mathbf{P}^n_K$, then $f(a_1), \ldots, f(a_m)$ are in general position and $f(\langle a_1, \ldots, a_m \rangle) = \langle f(a_1), \ldots, f(a_m) \rangle$.*

Proof. It suffices to consider $m \leqslant n+1$. If $m = 1$ the conclusion is just the definition of a collineation. So let $2 \leqslant m \leqslant n+1$ and assume by induction that the lemma has been verified for $m-1$ points. We write $f(a) = \hat{a}$. Since $f(\langle a_1, \ldots, a_{m-1}\rangle) = \langle \hat{a}_1, \ldots, \hat{a}_{m-1}\rangle$ and f is injective, it follows that $\hat{a}_m \notin \langle \hat{a}_1, \ldots, \hat{a}_{m-1}\rangle$ and thus $\hat{a}_1, \ldots, \hat{a}_m$ are in general position. The second conclusion follows from the fact that $\langle \hat{a}_1, \ldots, \hat{a}_m\rangle$ is the union of lines $\langle \hat{a}_m, b\rangle$, where b runs through the points of $\langle \hat{a}_1, \ldots, \hat{a}_{m-1}\rangle$. $\square$

LEMMA (b). *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation. If there exists a line $L \in \mathscr{L}^n_K$ such that $f|_L : L \to f(L)$ is projective-linear, then $f \in \mathrm{PGL}(n+1, K)$.*

Proof. Let $\tilde{e}_j = (0, \ldots, \overset{j\text{-th}}{1}, \ldots, 0) \in K^{n+1}$, $0 \leqslant j \leqslant n$, $\tilde{\delta} = \widetilde{e_0} + \cdots + \widetilde{e_n}$, and let $e_0, \ldots, e_n, \delta$ be the corresponding points in $\mathbf{P}^n_K$. *Let* $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be as in the hypothesis; we can assume without loss of generality that $f|_{\langle e_0, e_1\rangle}$ is projective-linear. By Lemma (a), the points $f(e_0), \ldots, f(e_n), f(\delta)$ are in general position. Choose representatives $\widetilde{f(e_0)}, \ldots, \widetilde{f(e_n)}, \widetilde{f(\delta)}$ in $K^{n+1}\backslash\{0\}$ of $f(e_0), \ldots, f(e_n), f(\delta)$ respectively. Let $\lambda_j \in K\backslash\{0\}$ $(0 \leqslant j \leqslant n)$ be given by $\sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$, and let $T \in GL(n+1, K)$ be given by $T(\tilde{e}_j) = \lambda_j \widetilde{f(e_j)}$. Then $T(\tilde{\delta}) = \sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$.

Let $\varphi = T^{-1} \circ f$. Thus the lemma is reduced to the following statement:

(A_n) *Let $\varphi: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation such that $\varphi|_{\langle e_0, e_1\rangle}$ is projective-linear, $\varphi(e_j) = e_j$ $(0 \leqslant j \leqslant n)$, and $\varphi(\delta) = \delta$. Then φ is the identity.*

We verify (A_n) by induction on n. For $n = 1$ the conclusion is immediate. So let $n \geqslant 2$ and assume (A_{n-1}). We write $\mathbf{P}^{n-1}_K = \langle e_0, \ldots, e_{n-1}\rangle$ and let $\delta' = (1 : \ldots : 1 : 0) \in \mathbf{P}^{n-1}_K$; thus $\langle e_n, \delta\rangle \cap \mathbf{P}^{n-1}_K = \{\delta'\}$. By Lemma (a), $\varphi(\mathbf{P}^{n-1}_K) = \mathbf{P}^{n-1}_K$ and thus $\varphi(\delta') = \delta'$. Hence by ($\mathrm{A}_{n-1}$), φ is the identity on $\mathbf{P}^{n-1}_K$. If a line $L \in \mathscr{L}^n_K$ contains a point $b \notin \mathbf{P}^{n-1}_K$ such that $\varphi(b) = b$, then $\varphi(L) = L$, since L must contain another fixed point of φ in $\mathbf{P}^{n-1}_K$. Let $a \in \langle e_0, e_n\rangle$, $a \neq e_0$, be arbitrary. Since $\{a\} = \langle a, \delta\rangle \cap \langle e_0, e_n\rangle$ and the points δ, e_n are fixed by φ, it follows that $\varphi(\langle a, \delta\rangle) = \langle a, \delta\rangle$ and $\varphi(\langle e_0, e_n\rangle) = \langle e_0, e_n\rangle$ and thus $\varphi(a) = a$. Finally, let $x \in \mathbf{P}^n_K\backslash\langle e_0, e_n\rangle$ be arbitrary. Since $\{x\} = \langle a, x\rangle \cap \langle e_n, x\rangle$, where a is as above and φ fixes a, e_n, it follows as before that $\varphi(x) = x$. $\square$

Proof of Proposition 1. Consider the usual embeddings $\mathbf{P}^1_K \subset \mathbf{P}^2_K \subset \mathbf{P}^n_K$. By Lemma (a), $f(\mathbf{P}^2_K)$ is a projective 2-plane. Hence there exists a projective linear map $T: f(\mathbf{P}^2_K) \to \mathbf{P}^2_K$ such that the map $f' = T \circ f|_{\mathbf{P}^2_K} : \mathbf{P}^2_K \to \mathbf{P}^2_K$ leaves the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and $(1:1:1)$ fixed. Then, for each $a \in K$, we can write $f'(1:a:0) = (1:\hat{a}:0)$, where $\hat{a} \in K$. We observe that the map $a \mapsto \hat{a}$ is an element of $\mathrm{Gal}(K)$. This follows from the fact that if $a, b \in K$, then $a - b$ and a/b can be constructed from the following "projective straightedge" constructions:

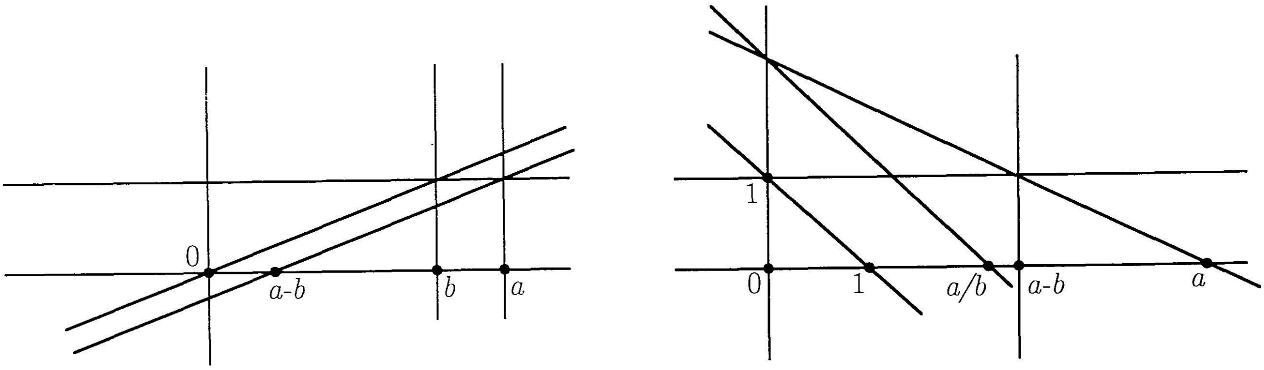

FIGURE 0

(Figure 0 shows the affine plane $K^2 \subset \mathbf{P}^2_K$.) Let $g \in \mathrm{Gal}(K)$ with $g(a) = \hat{a}$. Then $f' \circ g^{-1}|_{\mathbf{P}^1_K}$ is the identity map, and it follows that the map $f \circ g^{-1}|_{\mathbf{P}^1_K}: \mathbf{P}^1_K \to f(\mathbf{P}^1_K)$ is projective-linear. Therefore by Lemma (b), $f \circ g^{-1} = A' \in \mathrm{PGL}(n+1, K)$, and thus $f = A' \circ g = g \circ A$, where $A = g^{-1}A'g \in \mathrm{PGL}(n+1, K)$. $\square$

For a subset $U \subset \mathbf{P}^n_K$, we write

$$\mathscr{L}(U) = \{L \in \mathscr{L}^n_K : L \cap U \neq \emptyset\}\,.$$

We give the projective spaces $\mathbf{P}^n_{\mathbf{R}}, \mathbf{P}^n_{\mathbf{C}}$ and the Grassmannians $\mathscr{L}^n_{\mathbf{R}}, \mathscr{L}^n_{\mathbf{C}}$ the usual metric topologies. The main result of this section gives a condition for a local collineation to be projective-linear:

THEOREM 3. *Let U be a connected open set in $\mathbf{P}^n_K (n \geqslant 2)$, where K denotes either $\mathbf{R}$ or $\mathbf{C}$, and let $\mathscr{L}_0$ be an open subset of $\mathscr{L}(U)$ such that $\bigcup \mathscr{L}_0 \supset U$. Suppose that $f: U \to \mathbf{P}^n_K$ is a continuous injective map such that $f(L \cap U)$ is contained in a projective line for all $L \in \mathscr{L}_0$. Then there exists $A \in \mathrm{PGL}(n+1, K)$ such that*

(i) $f = A|_U$, *if* $K = \mathbf{R}$,

(ii) $f = A|_U$ *or* $\bar{f} = A|_U$, *if* $K = \mathbf{C}$.

The case $K = \mathbf{R}$ of Theorem 3 follows easily from Prenowitz's theorem [Pr, Theorem V], which provides a much stronger result for $n = 2$. (We include an elementary proof of the case $K = \mathbf{R}$ below.)

We begin by proving the following weaker form of Theorem 3:

LEMMA 4. *Let U be an open set in $\mathbf{P}_K^n (n \geqslant 2)$, where K denotes either $\mathbf{R}$ or $\mathbf{C}$, and let $f: U \to \mathbf{P}_K^n$ be a continuous injective map. If $f(L \cap U)$ is contained in a projective line for all $L \in \mathscr{L}(U)$, then the conclusion of Theorem 3 holds.*

Proof. Let $f: U \to \mathbf{P}_K^n$ be as in the statement of the lemma, and let $f(U) = \hat{U}$. We write $\hat{a} = f(a)$ for $a \in U$. Note that if three points a_1, a_2, a_3 of U are not collinear, then $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are not collinear, since otherwise the sets $f(\langle a_1, a_2 \rangle \cap U)$ and $f(\langle a_1, a_3 \rangle \cap U)$ would both be neighborhoods of a_1 in the line $\langle \hat{a}_1, \hat{a}_2 \rangle$ and hence f would not be injective. We also observe that if $L = \langle a, b \rangle$, where a, b are distinct points of U, then by hypothesis, $f(L \cap U) \subset \langle \hat{a}, \hat{b} \rangle$, and in fact we have $f(L \cap U) = \langle \hat{a}, \hat{b} \rangle \cap \hat{U}$. To verify this equality, let $\chi \in \langle \hat{a}, \hat{b} \rangle \cap \hat{U}$ be arbitrary and write $\chi = \hat{x}$, where $x \in U$. Since $\hat{a}, \hat{b}, \hat{x}$ are collinear, it follows from the above that x, a, b are collinear and thus $x \in L$.

We first consider the case $n = 2$. Choose a connected open set $U_0 \subset U$. Let $x \in \mathbf{P}_K^2$. We want to define $\hat{x} = \tilde{f}(x)$. Choose $a, b \in U_0$ such that a, b, x are not collinear. Let $\hat{L}_a, \hat{L}_b \in \mathscr{L}(\hat{U})$ be given by $f(\langle a, x \rangle \cap U) = \hat{L}_a \cap \hat{U}$, $f(\langle b, x \rangle \cap U) = \hat{L}_b \cap \hat{U}$. We define $\hat{x}(a, b) \in \mathbf{P}_K^2$ by

$$\hat{L}_a \cap \hat{L}_b = \hat{x}(a, b) .$$

(Note that $\hat{L}_a \neq \hat{L}_b$ since $\langle a, x \rangle \neq \langle b, x \rangle$ and f is injective.)

We observe that if $a' \in \langle a, x \rangle \cap U_0$, $b' \in \langle b, x \rangle \cap U_0$ with $a' \neq a$, $b' \neq b$, then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}, \hat{b}' \rangle .$$

In particular if $x \in U$, then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{x} \rangle \cap \langle \hat{b}, \hat{x} \rangle = \hat{x} .$$

STEP 1. *$\hat{x}(a, b)$ is independent of the choice of $a, b \in U_0$.*

We can assume by the above that $x \notin U$. Let $a \in U_0$ and let $b_0, b_1 \in U_0 \setminus \langle a, x \rangle$ be arbitrary. It suffices to show that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$.

We first consider the case $K = \mathbf{C}$. Let C be a real curve from b_0 to b_1 in $U_0 \backslash \langle a, x \rangle$. Let $\varepsilon > 0$, and suppose that b_2, b_3 are points in C such that $\operatorname{dist}(b_2, b_3) < \varepsilon$ (with respect to some metric on $\mathbf{P}^2_{\mathbf{C}}$ defining the usual topology). Choose $a', a'' \in \langle a, x \rangle \cap U_0$ with a, a', a'' distinct. Then let

$$b_3', b_3'', b_2', c, b_2''$$

be constructed (in the above order) as in Figure 1 below.

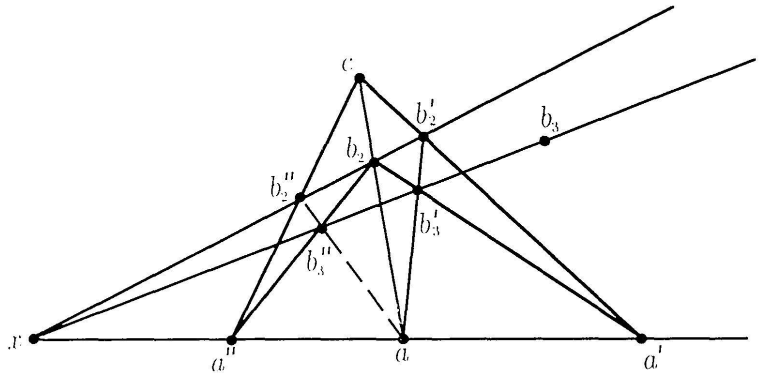

FIGURE 1

We claim that a, b_2'', b_3'' are collinear: Let $b_3^* = \langle a, b_2'' \rangle \cap \langle a'', b_2 \rangle$; to verify the claim, we must show that $b_3^* = b_3''$. By Desargues' Theorem [Co, 2.32; see Fig. 4.4a on p. 39] b_3', b_3^*, x are collinear and thus

$$b_3^* \in \langle b_3', x \rangle \cap \langle a'', b_2 \rangle = b_3'',$$

as desired.

We note that if $b_3 = b_2$, then

$$b_2 = b_3' = b_3'' = b_2' = c = b_2''.$$

Since C is compact, it follows that we can choose ε small enough so that all the labeled points in Figure 1 except x lie in U_0 whenever b_2, b_3 are points of C with $\operatorname{dist}(b_2, b_3) < \varepsilon$. Again by Desargues' Theorem, $\langle \hat{a}, \hat{a}' \rangle$, $\langle \hat{b}_2, \hat{b}_2' \rangle$ and $\langle \hat{b}_3', \hat{b}_3'' \rangle$ are coincident. Thus

$$\begin{aligned} \hat{x}(a, b_2) &= \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_2, \hat{b}_2' \rangle = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3', \hat{b}_3'' \rangle \\ &= \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3, \hat{b}_3' \rangle = \hat{x}(a, b_3). \end{aligned}$$

It follows that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$, which completes Step 1 for the case $K = \mathbf{C}$.

We now suppose that $K = \mathbf{R}$. (The proof must be modified for the case $K = \mathbf{R}$, since $U_0 \backslash \langle a, x \rangle$ may not be connected.) We may assume without loss of generality that the line segment

$$C \stackrel{\text{def}}{=} \{t b_0 + (1 - t) b_1 : 0 \leqslant t \leqslant 1\}$$

is contained in U_0. If $C \cap \langle a, x \rangle = \emptyset$, then we conclude that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$, by the proof for the case $K = \mathbf{C}$ above. On the other hand, if $C \cap \langle a, x \rangle = b'$, then

$$\hat{x}(b_0, a) = \hat{x}(b_0, b') = \hat{x}(b_0, b_1) = \hat{x}(b', b_1) = \hat{x}(a, b_1),$$

which completes Step 1 for the case $K = \mathbf{R}$.

We now write $\hat{x} = \hat{x}(a, b) = \tilde{f}(x)$ for all $x \in \mathbf{P}^2_K$.

STEP 2. *$\tilde{f}$ is a collineation.*

Let x, y, z be collinear. We must show that $\hat{x}, \hat{y}, \hat{z}$ are collinear. Choose collinear points $a, b, c \in U_0 \backslash \langle x, y \rangle$. Let a', b', c' be as in Figure 2 below. We note that if $a = b = c$, then $a' = b' = c' = a$. Thus we can choose distinct collinear $a, b, c \in U_0 \backslash \langle x, y \rangle$ such that a', b', c' are in U_0. By moving the line $\langle a, b \rangle$ slightly if necessary, we can assume further that $x, y, z \notin \langle a, b \rangle$, and hence a', b', c' are distinct. By Pappas' Theorem (see for example [Co, 4.41 and Fig. 4.4a]), a', b', c' are collinear. It further follows from the above that no four of the nine labeled points in Figure 2 are collinear. By the collinearity of f on U, the points $\hat{a}, \hat{b}, \hat{c}$ are collinear and distinct, and the same is true for $\hat{a}', \hat{b}', \hat{c}'$; furthermore, no four of the points $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}'$ are collinear. Hence $\hat{x}, \hat{y}, \hat{z}$ are distinct, and thus $\tilde{f}$ is injective. Applying Pappas' Theorem again (with $a, b, c, x, y, z, a', b', c'$ replaced by $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}', \hat{x}, \hat{y}, \hat{z}$, respectively), we conclude that $\hat{x}, \hat{y}, \hat{z}$ are collinear.

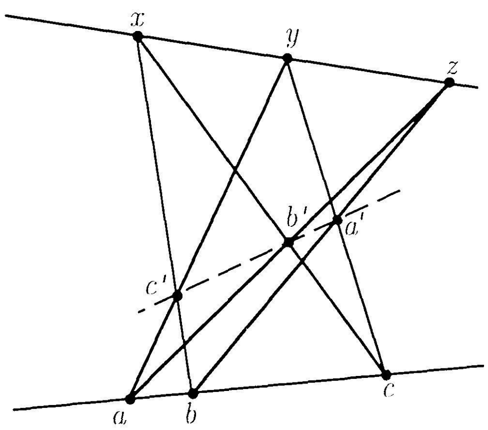

FIGURE 2

Finally, to show that $\tilde{f}$ is surjective, let $\chi \in \mathbf{P}^2_K$ be arbitrary. Choose points $\alpha, \alpha', \beta, \beta' \in \hat{U}_0 = f(U_0)$ such that $\chi = \langle \alpha, \alpha' \rangle \cap \langle \beta, \beta' \rangle$. The points $\alpha, \alpha', \beta, \beta'$ are the respective images of points $a, a', b, b' \in U_0$. If we set $x = \langle a, a' \rangle \cap \langle b, b' \rangle$, then $\chi = \hat{x}$.

Hence $\tilde{f}$ is a collineation. The case $n = 2$ then follows from Corollary 2.

STEP 3. *The proof for* $n > 2$.

Let $n > 2$. We easily see that f takes 2-planes in U to 2-planes in $\hat{U}$. Let $L \in \mathscr{L}(U)$ be arbitrary. By applying the case $n = 2$ to a projective 2-plane containing L, we see that $f|_{L \cap U}: L \cap U \to \hat{L} \cap \hat{U}$ is either projective-linear or anti-projective-linear. If $f|_{L \cap U}$ is anti-projective-linear for one L, it must be anti-projective-linear for all L (by the case $n = 2$), so by replacing f with $\bar{f}$ if necessary, we can assume that $f|_{L \cap U}$ is projective-linear for all $L \in \mathscr{L}(U)$. Now fix $a \in U$. For $x \in \mathbf{P}^n_K$, define $\hat{x} = T(x)$ where $T: \langle a, x \rangle \to \langle \hat{a}, \hat{x} \rangle$ is the projective-linear transformation extending $f|_{\langle a, x \rangle \cap U}$. By applying the case $n = 2$ to the plane determined by a, a', x (for an arbitrary point $a' \notin \langle a, x \rangle$), we see that $\hat{x}$ is independent of a. Thus we can define $\tilde{f}(x) = \hat{x}$. If x, y, z are collinear and $a \notin \langle x, y \rangle$, then the case $n = 2$ applied to the plane determined by a, x, y implies that $\hat{x}, \hat{y}, \hat{z}$ are collinear. The injectivity of $\tilde{f}$ similarly follows from the case $n = 2$. To show surjectivity, let $\chi \in \mathbf{P}^n_K$ be arbitrary, and choose a point $\alpha \in \langle \hat{a}, \chi \rangle \cap \hat{U} \setminus \{\hat{a}\}$. Then α is the image of a point $a' \in U$ and $\tilde{f}(\langle a, a' \rangle) = \langle \hat{a}, \alpha \rangle$. Hence $\chi \in \langle \hat{a}, \alpha \rangle \subset \text{image } \tilde{f}$.

Thus $\tilde{f}$ is a collineation. The conclusion of the lemma follows as before from Corollary 2. □

DEFINITION. A subset U of $\mathbf{P}^n_{\mathbf{R}}$ or $\mathbf{P}^n_{\mathbf{C}}$ is said to be *projectively convex* if $L \cap U$ is connected for all projective lines $L \in \mathscr{L}(U)$. (Note that if $U \subset \mathbf{R}^n \subset \mathbf{P}^n_{\mathbf{R}}$, then U is projectively convex if and only if U is convex.)

We use the following lemma to complete the proof of Theorem 3:

LEMMA 5. *Let U be a projectively convex, open set in $\mathbf{P}^n_K$, where K denotes either $\mathbf{R}$ or $\mathbf{C}$, and let $\mathscr{L}_0$ be an open subset of $\mathscr{L}(U)$ such that $\bigcup \mathscr{L}_0 \supset U$. Suppose that $f: U \to \mathbf{P}^n_K$ is a continuous injective map such that $f(L \cap U)$ is contained in a projective line for each $L \in \mathscr{L}_0$. Then $f(L \cap U)$ is contained in a projective line for every $L \in \mathscr{L}(U)$.*

Proof. We again write $\hat{p} = f(p)$, for $p \in U$. Let $L \in \mathscr{L}(U)$ be arbitrary, and let $x \in L \cap U$. Since $L \cap U$ is connected, it suffices to show that there is a neighborhood $V \subset U$ of x such that $\hat{x}, \hat{y}, \hat{z}$ are collinear whenever $y, z \in L \cap V$. Choose a line $L_x \in \mathscr{L}_0$ containing x. We can assume that $L_x \neq L$, since otherwise we are done. Choose $w \in L_x \cap U$, $w \neq x$. Next choose a neighborhood $V \subset U$ of x such that $\langle y, w \rangle \in \mathscr{L}_0$ for all $y \in V$.

Let $y, z \in L \cap V$. We must show that $\hat{x}, \hat{y}, \hat{z}$ are collinear. We can assume that x, y, z are distinct points. Choose $v \in L \cap V$ distinct from x, y, z (see Figure 3). Since $\langle v, w \rangle \in \mathscr{L}_0$, we can choose $a \in L_x \setminus \{x, w\}$ sufficiently close to w so that the line $L_a = \langle v, a \rangle \in \mathscr{L}_0$. Let $b = \langle y, w \rangle \cap L_a$, $c = \langle z, w \rangle \cap L_a$. By choosing a close enough to w, we can assume further that $a, b, c \in U$ and the six lines

$$\langle x, b \rangle, \ \langle x, c \rangle, \ \langle y, a \rangle, \ \langle y, c \rangle, \ \langle z, a \rangle, \ \langle z, b \rangle$$

are in $\mathscr{L}_0$. Let a', b', c' be as in Figure 3. Since all the points and lines of Figure 3 lie in a plane, we can use Desargues' Theorem to conclude that v, a', b', c' are collinear. Write $L' = \langle v, c' \rangle$; thus $a', b' \in L'$. Since a', b', c' (as well as b, c) converge to w as $a \to w$, by choosing a sufficiently close to w we can assume also that $a', b', c' \in U$ and $L' \in \mathscr{L}_0$. Since all the labeled points in Figure 3 lie in U and all the lines in Figure 3 except L are in $\mathscr{L}_0$, we conclude that the f-images of the points in Figure 3 lie in the plane determined by the image lines $\widehat{L_a}$ and $\widehat{L_x}$. We now apply Pappas' Theorem to the image to conclude (as in Step 2 of the proof of Lemma 4) that $\hat{x}, \hat{y}, \hat{z}$ are collinear. $\square$

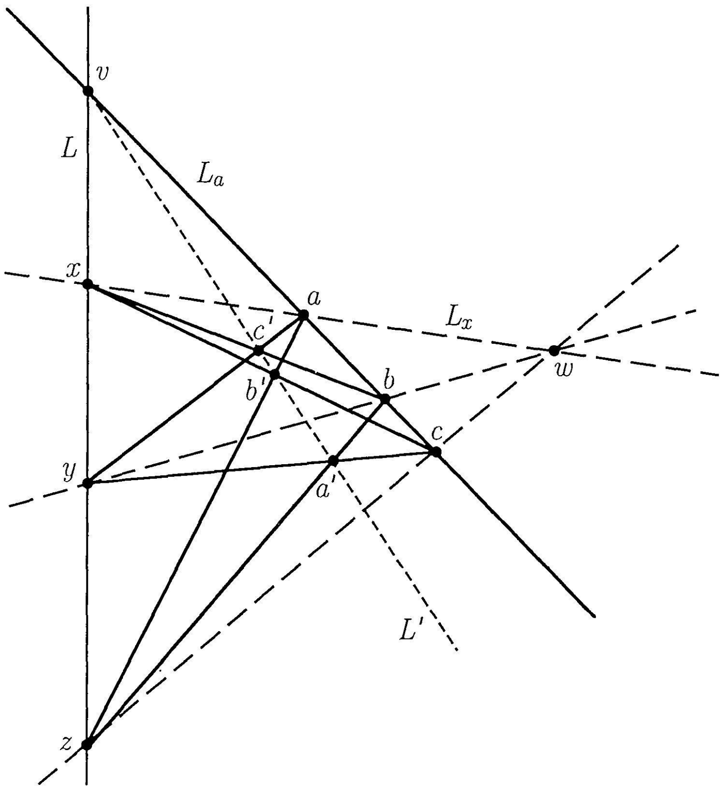

FIGURE 3

Proof of Theorem 3. Choose a sequence $\{U_1, U_2, \ldots\}$ of projectively convex, open subsets of U such that $U = \bigcup_{j=1}^{\infty} U_j$ and $U_1 \cup \cdots \cup U_j$ is connected for each $j \geqslant 1$. If $K = \mathbf{R}$, let $G = \mathrm{PGL}(n+1, \mathbf{R})$; if $K = \mathbf{C}$, let $G = \{e, \tau\} \cdot \mathrm{PGL}(n+1, \mathbf{C})$, where $\tau: \mathbf{P}^n_{\mathbf{C}} \to \mathbf{P}^n_{\mathbf{C}}$ is given by $\tau(z) = \bar{z}$ and e is the identity map. By Lemmas 5 and 4 applied to the restrictions $f|_{U_j}$, there are transformations $A_j \in G$ such that $f|_{U_j} = A_j|_{U_j}$. Since an element of G is uniquely determined by its values on a nonempty open subset of $\mathbf{P}^n_K$ and $(U_1 \cup \cdots \cup U_j) \cap U_{j+1} \neq \emptyset$, it follows by induction that $A_j = A_1$ for all j. Hence $f = A_1|_U$. $\square$

3. The Poincaré-Tanaka and Chern-Ji theorems

The Segre family $\mathscr{M}_{B_n}$ mentioned in the introduction has the projective analogue

$$\mathscr{M}^n_K = \{(z, w) \in \mathbf{P}^n_K \times \mathbf{P}^n_K : \sum_{j=0}^{n} z_j w_j = 0\} .$$

(In fact $\mathscr{M}^n_K$ is a compactification of $\mathscr{M}_{B_n}$; see the proof of Corollary 8.) We let $\pi_i: \mathbf{P}^n_K \times \mathbf{P}^n_K \to \mathbf{P}^n_K$ denote the projection to the i-th factor, for $i = 1, 2$. The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of $\mathbf{P}^n_K$ ($K = \mathbf{R}$ or $\mathbf{C}$) mapping $\mathscr{M}^n_K$ into itself must be projective-linear, or possibly anti-projective-linear (if $K = \mathbf{C}$):

THEOREM 6. *Let* $(a^1, a^2) \in \mathscr{M}^n_K$, *where* $K = \mathbf{R}$ *or* $\mathbf{C}$, $n \geqslant 2$. *Let* U_1, U_2 *be open sets in* $\mathbf{P}^n_K$ *containing* a^1, a^2 *respectively, and let* V_i *be the connected component of* $\pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$ *containing* a_i, *for* $i = 1, 2$. *If* $f_i: U_i \to \mathbf{P}^n_K$ $(i = 1, 2)$ *are continuous injective maps such that*

$$(f_1 \times f_2)(\mathscr{M}^n_K \cap U_1 \times U_2) \subset \mathscr{M}^n_K ,$$

then there exists $A \in \mathrm{PGL}(n+1, K)$ *such that*

(i) $f_1 = A$ *on* V_1 *and* $f_2 = {}^t A^{-1}$ *on* V_2, *if* $K = \mathbf{R}$,

(ii) *either* (i) *holds or* $\bar{f}_1 = A$ *on* V_1 *and* $\bar{f}_2 = {}^t A^{-1}$ *on* V_2, *if* $K = \mathbf{C}$.

REMARK. If the sets $\pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$ are connected, then $V_i = \pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$ and we have $\mathscr{M}^n_K \cap U_1 \times U_2 = \mathscr{M}^n_K \cap V_1 \times V_2$. In fact, if we assume that only one of the projections $\pi_1(\mathscr{M}^n_K \cap U_1 \times U_2)$ is connected, then by the uniqueness of A it follows that the conclusion of Theorem 6 holds with $V_i = \pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$, for $i = 1, 2$.

Proof of Theorem 6. For a point $w \in \mathbf{P}_K^n$ we write

$$w^{\perp} = \{z \in \mathbf{P}_K^n : z \cdot w = 0\} ,$$

where $z \cdot w = \sum_{j=0}^{n} z_j w_j$. For a subset $S \subset \mathbf{P}_K^n$ we also write

$$S^{\perp} = \{z \in \mathbf{P}_K^n : z \cdot w = 0 \ \forall w \in E\} .$$

We consider the collection of lines

$$\mathscr{L}_0 = \{L \in \mathscr{L}(V_1) : L^{\perp} \cap U_2 \neq \emptyset\} ,$$

which is open in $\mathscr{L}(V_1)$. If z is an arbitrary point of V_1, then by hypothesis we can choose $w \in U_2$ such that $(z, w) \in \mathscr{M}_K^n$. If we let L be any projective line in $w^{\perp}$ containing z, then $w \in L^{\perp} \cap U_2$ and hence $L \in \mathscr{L}_0$. Therefore $\bigcup \mathscr{L}_0 \supset V_1$.

Now let $L \in \mathscr{L}_0$ be arbitrary. We claim that we can choose points $w^1, \ldots, w^{n-1} \in L^{\perp} \cap U_2$, such that $f_2(w^1), \ldots, f_2(w^{n-1})$ are in general position: If $n = 2$, the claim is a tautology, so suppose $n \geqslant 3$. If the claim were false, then $f_2(L^{\perp} \cap U_2)$ must lie in a projective linear subspace $\mathbf{P}(E)$ of dimension $n - 3$ (where E is a linear subspace of K^{n+1} of dimension $n - 2$). But then f_2 would be a continuous injection from $(L^{\perp} \cap U_2)$, which has topological dimension $n - 2$ or $2n - 4$ (depending on whether K equals $\mathbf{R}$ or $\mathbf{C}$), into $\mathbf{P}(E)$, which has topological dimension $n - 3$ or $2n - 6$. This contradicts dimension theory.

Let $w^1, \ldots, w^{n-1} \in L^{\perp} \cap U_2$, such that $f_2(w^1), \ldots, f_2(w^{n-1})$ are in general position, as above. By moving the points slightly if necessary, we can assume also that $w^1, \ldots, w^{n-1}$ are in general position, and hence $L = \langle w^1, \ldots, w^{n-1} \rangle^{\perp}$. We note that by hypothesis, $f_1(w^{\perp} \cap U_1) \subset f_2(w)^{\perp}$ for all $w \in U_2$. Therefore

$$f_1(L \cap U_1) = \bigcap_{j=1}^{n-1} f_1(w^{j\perp} \cap U_1) \subset \bigcap_{j=1}^{n-1} f_2(w^j)^{\perp}$$

$$= \langle f_2(w^1), \ldots, f_2(w^{n-1}) \rangle^{\perp} \in \mathscr{L}_K^n(U_1) .$$

Let G be the group of projective-linear, and if $K = \mathbf{C}$, anti-projective linear, transformations of $\mathbf{P}_K^n$ as in the proof of Theorem 3. By Theorem 3, there exists $A \in G$ such that $f_1 = A$ on V_1; similarly, there exists $B \in G$ such that $f_2 = B$ on V_2. By replacing $f_1 \times f_2$ with $\bar{f}_1 \times \bar{f}_2$ if necessary, we can assume that $A \in \mathrm{PGL}(n + 1, K)$. We now show that $B = {}^tA^{-1}$: Let M be the connected component of $\mathscr{M}_K^n \cap U_1 \times U_2$ containing (a^1, a^2). Fix a point $w \in \pi_2(M) \subset V_2$, and choose $z^1, \ldots, z^n \in w^{\perp} \cap V_1$ in general position. Then $(Az^j, Bw) = (f_1(z^j), f_2(w)) \in \mathscr{M}_K^n$ since $(z^j, w) \in \mathscr{M}_K^n$, and thus

$$0 = Az^j \cdot Bw = z^j \cdot {}^tABw \,,$$

for $j = 1, \ldots, n$. Therefore ${}^tABw \in w^{\perp\perp} = \{w\}$. Since w is an arbitrary point of $\pi_2(M)$ and since elements of G are uniquely determined by their values on the open set $\pi_2(M)$, it follows that ${}^tAB$ is the identity $e \in G$, and therefore $B = {}^tA^{-1} \in \mathrm{PGL}(n+1, K)$. $\square$

COROLLARY 7 (Chern-Ji [CJ, Theorem 2]). *Suppose* $U, \hat{U}, V, \hat{V}$ *are connected open sets in* $\mathbf{P}^n_{\mathbf{C}}$ *such that* $\mathcal{M}^n_{\mathbf{C}} \cap U \times V \neq \emptyset$. *If* $f: U \to \hat{U}, g: V \to \hat{V}$ *are biholomorphic maps such that*

$$(f \times g)\,(\mathcal{M}^n_{\mathbf{C}} \cap U \times V) \subset \mathcal{M}^n_{\mathbf{C}} \,,$$

then f *and* g *are restrictions of elements of* $\mathrm{PGL}(n+1, \mathbf{C})$.

We conclude this paper by demonstrating how the following theorem of Poincaré and Tanaka is obtained from Corollary 7.

COROLLARY 8 (Poincaré-Tanaka Theorem) [Po], [Ta]. *Let* B_n *denote the unit ball in* $\mathbf{C}^n, n \geqslant 2$. *Suppose that* U *is a connected open set in* $\mathbf{C}^n$ *such that* $U \cap \partial B_n \neq \emptyset$. *If* $f: U \to \mathbf{C}^n$ *is a nonconstant holomorphic map such that* $f(U \cap \partial B_n) \subset \partial B_n$, *then* $f|_{U \cap B_n}$ *extends to an automorphism of* B_n.

Proof. By an elementary argument given by H. Alexander ([A], p. 250]), we can assume that the Jacobian matrix of f is nonsingular at some point $z_0 \in U \cap \partial B_n$. (We shall give Alexander's argument later.) By replacing U with a neighborhood of z_0, we can assume that f is injective. Let $\tau: \mathbf{C}^n \to \mathbf{C}^n$ be the conjugation $z \mapsto \bar{z}$. Let $V = \tau(U)$ and consider the holomorphic map $g = \tau \circ f \circ \tau: V \to \mathbf{C}^n$. We let $\hat{U} = f(U)$, $\hat{V} = g(V) = \tau(\hat{U})$ so that the maps $f: U \to \hat{U}$, $g: V \to \hat{V}$ are biholomorphic. We let ψ denote the function on $\mathbf{C}^n \times \mathbf{C}^n$ given by $\psi(z, w) = \sum_{j=1}^n z_j w_j - 1$ and we consider the "Segre family"

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \psi(z, w) = 0\} \,.$$

Let $S: \mathbf{C}^n \to \mathbf{C}^{2n}$ be given by $S(z) = (z, \bar{z})$, so that $S^{-1}(\mathcal{M}_{B_n}) = \partial B_n$ and $S \circ f = (f \times g) \circ S$. Let $\Omega = U \times V$ and $N = S(\partial B_n) = \mathcal{M}_{B_n} \cap S(\mathbf{C}^n)$. Then

$$(f \times g)\,(\Omega \cap N) = S \circ f(U \cap \partial B_n) \subset S(\partial B_n) = N \subset \mathcal{M}_{B_n} \,.$$

Choose a point $z_0 \in U \cap \partial B_n$; then $(z_0, \bar{z}_0) \in \Omega \cap N$. Since $\psi \circ (f \times g)$ vanishes on $\Omega \cap N$ and N is a totally real submanifold of (real) dimension $2n - 1$ in $\mathcal{M}_{B_n}$, it follows that $\psi \circ (f \times g)$ vanishes on the connected component of $\Omega \cap \mathcal{M}_{B_n}$ containing $(z_0, \bar{z}_0)$. After shrinking U if necessary, we can assume that $\psi \circ (f \times g)$ vanishes on $\Omega \cap \mathcal{M}_{B_n}$ and thus $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$. We consider the embedding $\iota \times \iota : \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}^n_{\mathbf{C}} \times \mathbf{P}^n_{\mathbf{C}}$ given by $\iota(z_1, \ldots, z_n) = (\sqrt{-1} : z_1 : \ldots : z_n)$, which maps $\mathcal{M}_{B_n}$ onto a (dense open) subset of $\mathcal{M}^n_{\mathbf{C}}$. By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1} : \iota(U) \to \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1} : \iota(V) \to \iota(\hat{V}),$$

there exists $A \in \mathrm{PGL}(n+1, \mathbf{C})$ such that $\tilde{f} = A|_{\iota(U)}$. Thus f extends to the fractional linear map $\iota^{-1} \circ A \circ \iota$, which gives an automorphism of B_n.

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map f must be nonsingular at some point of $U \cap \partial B_n$. We begin by observing that $f^{-1}(\partial B_n)$ is nowhere dense. Indeed, suppose on the contrary that $f^{-1}(\partial B_n)$ contains a connected open set U_0 and assume without loss of generality that $f(z_0) = (1, 0, \ldots, 0)$ for some point $z_0 \in U_0$. Then by the maximum principle, $f_1 \equiv 1$ and hence $f \equiv (1, 0, \ldots, 0)$ on U_0 and thus on U, contradicting the assumption that f is nonconstant. Now suppose on the contrary that the Jacobian determinant of f vanishes identically on $U \cap \partial B_n$. Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on U. As a consequence, the fibers of f contain no isolated points. Assume without loss of generality that $(1, 0, \ldots, 0) \in U$ and choose $r < 1$ such that the spherical cap $W := \{z \in B_n : \mathrm{Re}\, z_1 > r\}$ is contained in U. Choose a point $p \in W$ such that $f(p) \notin \partial B_n$. Let A be the connected component of $f^{-1}(f(p)) \cap W$ that contains p; A is an analytic subvariety of W of positive dimension. Furthermore $\bar{A} \backslash A \subset \{z \in \mathbf{C}^n : \mathrm{Re}\, z_1 = r\}$. By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function $\varphi : A \to \mathbf{C}$ given by $\varphi(z) = \exp z_1$, we conclude that φ is constant and thus $\bar{A} \backslash A = \emptyset$ so that A is a compact subvariety of W of positive dimension, which is impossible. □

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