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$$\begin{array}{ccc} H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W . \\ p_k^* \downarrow & & \downarrow 1 \otimes p_k^* \\ H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W . \end{array}$$

Since p_k^* acts by k^q on $H^q(T)$, (6.1) implies that $H^n(G)_q \simeq [H^{n-q}(G/T) \otimes H^q(T)]^W$, and (6.3) gives the dimension of the latter space.

(6.6) This last interpretation of the bigrading shows that it is natural in the following sense. Suppose $f: K \to G$ is a homomorphism between two compact connected Lie groups. Since f commutes with the power maps P_k on G and K, the cohomology map f^* sends $H^n(G)_q$ to $H^n(K)_q$. Suppose for example that K is a closed connected subgroup of G and f is the inclusion map. Choose, as we may, a maximal torus T of G such that $S := T \cap K$ is a maximal torus of K. The restriction map $H(G) \to H(K)$ becomes, via (6.1), the map $[H(G/T) \otimes H(T)]^W \to [H(K/S) \otimes H(S)]^{W_K}$ induced by restriction on each factor, where W_K is the Weyl group of S in K.

(6.7) We close with the homology interpretation of (6.1), which says the homology map ψ_* induced by ψ is surjective. It follows that the homology of G is spanned by the cycles $[\psi(\bar{X}_w, T_I)] = \{gtg^{-1}: gT \in \bar{X}_w, t \in T_I\}$. Here $w \in W$, X_w is the Schubert cell (see (5.2)) and $T_I = \prod_{i \in I} T_i$, where $T = T_1 \times \cdots \times T_l$, with each $T_i \simeq S^1$. Using the results in [BGG], one can explicitly write down the action of W on $H_*(G/T)$ in terms of the Schubert cell basis, and this leads, in principle, to the linear relations in $H_*(G)$ satisfied by the cycles $[\psi(\bar{X}_w, T_I)]$.

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