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where, as in (3.3),  $\partial_i$  is the derivation of  $\mathcal{S}$  extending the functional  $\lambda \mapsto \lambda(H_i)$ . We have a perfect pairing

$$\mathcal{D} \otimes \mathcal{S} \rightarrow \mathbf{R}$$

given by  $(D, f) = (Df)(0)$ . Since the pairing is perfect, something in degree  $\nu$  must pair nontrivially with  $\Pi$ . Since an irreducible  $W$ -module can only pair nontrivially with its dual, and the self-dual character  $\varepsilon$  occurs with multiplicity one in  $\mathcal{D}^\nu$ , afforded by  $\partial_1 \cdots \partial_\nu$ , we must have  $\partial_1 \cdots \partial_\nu \Pi \neq 0$ , so  $c(\Pi) \neq 0$ .

Observe that  $\partial_1 \cdots \partial_\nu$  is analogous to the fundamental class of  $G/T$ , and the pairing is essentially that between homology and cohomology. We further remark that in fact all irreducible representations of  $W$  are defined over the rational numbers, hence they are all self dual. This is a consequence of Springer's cohomological construction of  $W$ -modules [Sp].

Returning again to our task, we now inductively assume that  $c: \mathcal{H}^k \rightarrow H^{2k}(G/T)$  is injective for  $k \leq \nu$ , and let  $V = \mathcal{H}^{k-1} \cap \ker c$ . Note that  $V$  is preserved by  $W$  since  $c$  is  $W$ -equivariant. The sign character does not occur in  $\mathcal{H}^{k-1}$ , so there is a positive root  $\alpha$  whose corresponding reflection  $s_\alpha$  does not act by  $-I$  on  $V$ . Decompose  $V = V_+ \oplus V_-$  according to the eigenspaces of  $s_\alpha$ . If  $V \neq 0$  then  $V_+ \neq 0$ , so take  $f \in V_+$ . Now  $c(\alpha f) = c(\alpha)c(f) = 0$ , and  $\alpha f$  is in degree  $k$ , so we must have  $\alpha f \in \mathcal{S}$  by the induction hypothesis. Let  $h_1, \dots, h_{|W|}$  be a basis of  $\mathcal{H}$  with  $h_1, \dots, h_r$   $s_\alpha$ -skew and the rest  $s_\alpha$  invariant. By Chevalley's theorem (3.2), we can write  $\alpha f = \sum h_i \sigma_i$  with  $\sigma_i$   $W$ -invariant of positive degree. Since  $\alpha f$  is  $s_\alpha$ -skew, the sum only goes up to  $r$ . Now for  $i \leq r$ , the polynomial  $h_i$  must vanish on  $\ker \alpha$ , hence can be written  $h_i = \alpha h'_i$  for some  $h'_i \in \mathcal{S}$ . But then  $f = \sum_{i=1}^r h'_i \sigma_i \in \mathcal{S}$ . Since  $f$  is supposed to be harmonic, we must have  $f = 0$ . Hence  $c$  is injective on  $\mathcal{H}$ , and the proof of Borel's theorem is complete.  $\square$

## 6. THE COHOMOLOGY OF A LIE GROUP

We now have all the ingredients for our proof. Consider the map  $\psi: G/T \times T \rightarrow G$  given by  $\psi(gT, t) = gtg^{-1}$ . The Weyl group  $W$  acts on  $T$  by conjugation and on  $G/T$  by  $w \cdot gT = gn^{-1}T$ , where  $w = nT$ . Hence  $W$  acts on  $H(G/T \times T) = H(G/T) \otimes H(T)$ . Since  $\psi(gn^{-1}T, wtw^{-1}) = \psi(gT, t)$ , it follows that the induced map  $\psi^*$  on cohomology maps  $H(G)$  to  $[H(G/T) \otimes H(T)]^W$ . Though we prefer to have it in this form, the latter group could be thought of as the cohomology of the quotient of  $G/T \times T$

by the action of  $W$ , and this quotient has a natural interpretation. As in the introduction, let  $M$  be the set of pairs  $(g, T')$  where  $T'$  is a maximal torus in  $G$  containing  $g \in G$ . The map  $G/T \times_W T \rightarrow M$  sending  $(gT, t) \bmod(W)$  to  $(gtg^{-1}, gTg^{-1})$  is a diffeomorphism.

PROPOSITION (6.1). *The map induced by  $\psi$  on cohomology is an isomorphism of graded rings*

$$\psi^*: H(G) \xrightarrow{\cong} [H(G/T) \otimes H(T)]^W.$$

*Proof.* We compute the derivative  $(d\psi)_{(gT, t)}$  at a point  $(gT, t) \in G/T \times T$ . For each point  $gT \in G/T$ , we identify  $\mathfrak{m}$  with the tangent space  $T_{gT}(G/T)$  by letting  $X \in \mathfrak{m}$  correspond to the initial tangent vector  $X_{gT}$  of the path  $s \mapsto g(\exp sX)T$  in  $G/T$ . Similarly, an element  $X \in \mathfrak{g}$  (resp.  $H \in \mathfrak{t}$ ) corresponds to a tangent vector  $X_g \in T_g(G)$  (resp.  $H_t \in T_t(T)$ , for  $t \in T$ ).

Then

$$\begin{aligned} (d\psi)_{gT, t}(X_{gT}, 0) &= \frac{d}{ds} g(\exp sX)t(\exp -sX)g^{-1} \Big|_{s=0} \\ &= \frac{d}{ds} gtg^{-1} [\exp sAd(gt^{-1})X] [\exp -sAd(g)X] \Big|_{s=0} \\ &= \frac{d}{ds} gtg^{-1} [X + sAd(g)(Ad(t^{-1}) - 1)X + O(s^2)] \Big|_{s=0} \\ &= [Ad(g)(Ad(t^{-1}))X]_{gtg^{-1}}. \end{aligned}$$

Similarly, we find, for  $H \in \mathfrak{t}$ , that

$$(d\psi)_{gT, t}(0, H_t) = [Ad(g)H]_{gtg^{-1}}.$$

Hence, under the identifications,  $(d\psi)_{(gT, t)}$  is the map

$$(Ad(t^{-1}) - I)_{\mathfrak{m}} \oplus I: \mathfrak{m} \oplus \mathfrak{t} \rightarrow \mathfrak{m} \oplus \mathfrak{t} = \mathfrak{g}.$$

Here the subscript  $\mathfrak{m}$  means we view  $Ad(t^{-1}) - I$  as a map from  $\mathfrak{m}$  to itself. Now  $G$  being compact and connected, we must have  $\det Ad(t) = 1$ , so

$$\det (d\psi)_{(gT, t)} = \det (I - Ad(t))_{\mathfrak{m}}.$$

(Actually,  $\mathfrak{m}$  is always even-dimensional as we have seen, so there is no need to reverse the subtraction).

We compute the degree of  $\psi$  by finding a regular value. Let  $t_0$  be a generic element in  $T$ , as in (2.3). Consider  $\psi^{-1}(t_0) = \{(gT, t): gtg^{-1} = t_0\}$ . It turns out that any two elements of  $T$  conjugate in  $G$  must be conjugate by an element

of  $W$ . (In  $U(n)$ , two diagonal matrices with the same set of eigenvalues must be conjugate by a permutation matrix.) It follows easily then that

$$\psi^{-1}(t_0) = \{(wT, wt_0 w^{-1}) : w \in W\}.$$

We next show that  $\psi$  preserves orientation at each point in  $\psi^{-1}(t_0)$ . The eigenvalues of  $Ad(t_0)$  in  $\mathfrak{m}$  are complex conjugate pairs  $z, \bar{z}$ , where  $|z| = 1, z \neq 1$ . Hence  $|1 - z||1 - \bar{z}| = 2(1 - \operatorname{Re}(z)) > 0$ , so  $\det(I - Ad(t_0))_{\mathfrak{m}} > 0$ .

At this point we know the degree of  $\psi$  is  $\deg \psi = |W| \neq 0$ . By Poincaré duality, any smooth map between compact manifolds of the same dimension is injective on cohomology as soon as it has nonzero degree. Hence  $\psi^* : H(G) \rightarrow [H(G/T) \times H(T)]^W$  is injective. We finish the proof of (6.1) by showing that both sides have the same dimension.

For this we use, three times, the following basic principle. Let  $K$  be a compact group (here  $K$  will be  $G, T$  or  $W$ ). Let  $dk$  be the left invariant Haar measure on  $K$  with total mass one. Let  $V$  be a finite dimensional real vector space, and  $\rho : K \rightarrow GL(V)$  a continuous group homomorphism. Then the space  $V^K$  of vectors fixed by all  $\rho(k), k \in K$ , has dimension

$$\dim V^K = \int_K \operatorname{trace} \rho(k) dk.$$

To compute this integral over  $G$ , we must exploit further the computation of  $d\psi$ . Let  $\omega_G, \omega_T, \omega_{G/T}$  be the unique invariant (under left translations by  $G, T$ , and  $G$  respectively) differential forms of top degree whose integral over the respective manifold is one. The the pull-back formula for integration gives

$$\int_G f \omega_G = \frac{1}{\deg \psi} \int_{G/T \times T} f \circ \psi(gT, t) |\det(d\psi)_{(gT, t)}| \omega_{G/T} \wedge \omega_T,$$

where the determinant is computed with respect to bases spanning parallelograms of unit volume with respect to the appropriate forms. Taking  $f$  to be invariant under conjugation by  $G$ , we arrive at the famous Weyl integration formula:

$$\int_G f \omega_G = \frac{1}{|W|} \int_T f(t) \det(I - Ad(t))_{\mathfrak{m}} \omega_T.$$

Expand the function  $t \mapsto \det(I - Ad(t))_{\mathfrak{m}}$  in a sum of characters of  $T$ :  $n_0 \chi_0 + n_1 \chi_1 + \cdots + n_k \chi_k$ . Here  $\chi_0$  is the trivial character of  $T$ ,

appearing  $n_0$  times, and for  $i > 0$  each  $\chi_i$  is a nontrivial character appearing  $n_i$  times. Taking for  $f$  the constant function equal to one, and applying the basic principle of invariants to  $T$ , we find  $n_0 = |W|$ .

Taking for  $f$  the function  $f(g) = \det(I + Ad(g))$ , i.e., the trace of  $Ad(g)$  acting on  $\Lambda\mathfrak{g}$ , we find, using the Cartan-de Rham isomorphism (4.3), that

$$\begin{aligned} \dim H(G) &= \dim(\Lambda\mathfrak{g})^G = \int_G \det(I + Ad(g)) \omega_G \\ &= \frac{1}{|W|} \int_T \det(I + Ad(t)) \det(I - Ad(t))_m \omega_T \\ &= \frac{2^{\dim T}}{|W|} \int_T \det(I - Ad(t^2))_m \omega_T. \end{aligned}$$

Now the squaring map on  $T$  is surjective, so the square of a nontrivial character of  $T$  is still nontrivial. Hence the trivial character again appears with multiplicity  $|W|$  in the expansion of  $\det(I - Ad(t^2))_m$ . This multiplicity is the value of the integral, so  $\dim H(G) = 2^{\dim T} = 2^l$ .

On the other hand, we saw in (5.3) that the trace of  $w \in W$  acting on  $H(G/T)$  is  $|W|$  if  $w = 1$ , zero otherwise. Applying the invariance formula one more time, we find that  $\dim [H(G/T) \otimes H(T)]^W = 2^l$  as well, completing the proof of (6.1).  $\square$

We now have the main result

(6.2) THEOREM. *The cohomology ring  $H(G)$  with real coefficients is a bigraded exterior algebra with generators in bi-degrees  $(2m_i, 1)$ , for  $1 \leq i \leq l$ .*

*Proof.* By (6.1) and (5.4), we have

$$H(G) \simeq [H(G/T) \otimes H(T)]^W \simeq [\mathcal{H}_{(2)} \otimes \Lambda]^W,$$

and by (3.8), the latter space is an exterior algebra with generators in degrees  $(2m_i, 1)$ , for  $1 \leq i \leq l$ .  $\square$

Moreover, from the multiplicity formula (3.8), the dimensions of the bi-graded pieces are given in terms of the exponents as follows

(6.3) COROLLARY. For each  $q \geq 0$ , we have

$$\sum_{n=0}^{\dim G} \dim [H^{n-q}(G/T) \otimes H^q(T)]^W u^n = u^q s_q(u^{2m_1}, \dots, u^{2m_l}).$$

(6.4) We give two interpretations of the bigrading. First, we follow [L] and consider the spectral sequence of the fibration  $G \rightarrow G/T$ , which has  $E_2$ -term

$$E_2^{p,q} = H^p(G/T) \otimes H^q(T),$$

and converges to  $H(G)$ . This spectral sequence does not degenerate at  $E_2$ , but it has a spectral subsequence which does degenerate, and still converges to  $H(G)$ .

To see this we again consider the Weyl group action. More precisely,  $N$  acts by automorphisms of the fibration  $G \rightarrow G/T$ , which in turn induce automorphisms of each term in the spectral sequence, commuting with the differentials. On  $E_2^{p,q} = H^p(G/T) \otimes H^q(T)$ , the action of  $N$  factors through  $W$  and is the same as that considered above. Thus we have representations of  $W$  on the spaces  $E_2^{p,q}$ , hence on each  $E_r^{p,q}$  for  $r \geq 2$ .

For each  $p, q, r$  we decompose  $E_r^{p,q} = (E_r^{p,q})^W \oplus (E_r^{p,q})_W$ , where the subscript  $W$  indicates a  $W$ -stable complement to the invariants. Each of the latter two spaces is a spectral subsequence, and since  $E_\infty^{p,q}$  is a subquotient of  $H^{p+q}(G)$  and  $N$  acts trivially on  $H(G)$  (because  $G$  is connected), we must have  $(E_\infty^{p,q})_W = 0$ . On the other hand,  $(E_\infty^{p,q})^W$  is a subquotient of  $(E_2^{p,q})^W = [H^p(G/T) \otimes H^q(T)]^W$ , so we have

$$\begin{aligned} 2^l = \dim H(G) &= \sum_{p,q} \dim (E_\infty^{p,q})^W \leq \sum_{p,q} \dim (E_2^{p,q})^W \\ &= \sum_q \dim [H(G/T) \otimes \Lambda^q]^W = 2^l. \end{aligned}$$

It follows that  $\dim (E_\infty^{p,q})^W = \dim (E_2^{p,q})^W$  for all  $p, q$ , so the spectral subsequence of  $W$ -invariants degenerates at  $(E_2)^W$ , and (6.1) is proved again.

(6.5) The significance of the bigrading on  $H(G)$  can be seen in yet another way, inspired by [GHV]. We consider, for a fixed integer  $k \neq 1$ , the  $k^{\text{th}}$ -power maps  $x \mapsto x^k$ , denoted  $p_k$  and  $P_k$ , on  $T$  and  $G$ , respectively. It is shown in [GHV] that the Lefschetz number of  $P_k$  equals that of  $p_k$ , namely  $(1-k)^l$ . Let  $H^n(G)_q$  be the  $k^q$ -eigenspace of  $P_k^*$  acting on  $H^n(G)$ . It is further shown in [GHV] that  $\sum_n \dim H^n(G)_q = \binom{l}{q}$ . We can refine this by computing each  $\dim H^n(G)_q$  separately. Consider the commutative diagram

$$\begin{array}{ccc}
 H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W . \\
 p_k^* \downarrow & & \downarrow 1 \otimes p_k^* \\
 H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W .
 \end{array}$$

Since  $p_k^*$  acts by  $k^q$  on  $H^q(T)$ , (6.1) implies that  $H^n(G)_q \simeq [H^{n-q}(G/T) \otimes H^q(T)]^W$ , and (6.3) gives the dimension of the latter space.

(6.6) This last interpretation of the bigrading shows that it is natural in the following sense. Suppose  $f: K \rightarrow G$  is a homomorphism between two compact connected Lie groups. Since  $f$  commutes with the power maps  $P_k$  on  $G$  and  $K$ , the cohomology map  $f^*$  sends  $H^n(G)_q$  to  $H^n(K)_q$ . Suppose for example that  $K$  is a closed connected subgroup of  $G$  and  $f$  is the inclusion map. Choose, as we may, a maximal torus  $T$  of  $G$  such that  $S := T \cap K$  is a maximal torus of  $K$ . The restriction map  $H(G) \rightarrow H(K)$  becomes, via (6.1), the map  $[H(G/T) \otimes H(T)]^W \rightarrow [H(K/S) \otimes H(S)]^{W_K}$  induced by restriction on each factor, where  $W_K$  is the Weyl group of  $S$  in  $K$ .

(6.7) We close with the homology interpretation of (6.1), which says the homology map  $\psi_*$  induced by  $\psi$  is surjective. It follows that the homology of  $G$  is spanned by the cycles  $[\psi(\bar{X}_w, T_I)] = \{gtg^{-1}: gT \in \bar{X}_w, t \in T_I\}$ . Here  $w \in W$ ,  $X_w$  is the Schubert cell (see (5.2)) and  $T_I = \prod_{i \in I} T_i$ , where  $T = T_1 \times \cdots \times T_l$ , with each  $T_i \simeq S^1$ . Using the results in [BGG], one can explicitly write down the action of  $W$  on  $H_*(G/T)$  in terms of the Schubert cell basis, and this leads, in principle, to the linear relations in  $H_*(G)$  satisfied by the cycles  $[\psi(\bar{X}_w, T_I)]$ .

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