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where, as in (3.3), ∂_i is the derivation of \mathcal{S} extending the functional $\lambda \mapsto \lambda(H_i)$. We have a perfect pairing

$$\mathcal{D} \otimes \mathcal{S} \rightarrow \mathbf{R}$$

given by $(D, f) = (Df)(0)$. Since the pairing is perfect, something in degree v must pair nontrivially with Π . Since an irreducible W -module can only pair nontrivially with its dual, and the self-dual character ε occurs with multiplicity one in \mathcal{D}^v , afforded by $\partial_1 \cdots \partial_v$, we must have $\partial_1 \cdots \partial_v \Pi \neq 0$, so $c(\Pi) \neq 0$.

Observe that $\partial_1 \cdots \partial_v$ is analogous to the fundamental class of G/T , and the pairing is essentially that between homology and cohomology. We further remark that in fact all irreducible representations of W are defined over the rational numbers, hence they are all self dual. This is a consequence of Springer's cohomological construction of W -modules [Sp].

Returning again to our task, we now inductively assume that $c: \mathcal{H}^k \rightarrow H^{2k}(G/T)$ is injective for $k \leq v$, and let $V = \mathcal{H}^{k-1} \cap \ker c$. Note that V is preserved by W since c is W -equivariant. The sign character does not occur in \mathcal{H}^{k-1} , so there is a positive root α whose corresponding reflection s_α does not act by $-I$ on V . Decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α . If $V \neq 0$ then $V_+ \neq 0$, so take $f \in V_+$. Now $c(\alpha f) = c(\alpha)c(f) = 0$, and αf is in degree k , so we must have $\alpha f \in \mathcal{I}$ by the induction hypothesis. Let $h_1, \dots, h_{|W|}$ be a basis of \mathcal{H} with h_1, \dots, h_r s_α -skew and the rest s_α invariant. By Chevalley's theorem (3.2), we can write $\alpha f = \sum h_i \sigma_i$ with σ_i W -invariant of positive degree. Since αf is s_α -skew, the sum only goes up to r . Now for $i \leq r$, the polynomial h_i must vanish on $\ker \alpha$, hence can be written $h_i = \alpha h'_i$ for some $h'_i \in \mathcal{S}$. But then $f = \sum_{i=1}^r h'_i \sigma_i \in \mathcal{I}$. Since f is supposed to be harmonic, we must have $f = 0$. Hence c is injective on \mathcal{H} , and the proof of Borel's theorem is complete. \square

6. THE COHOMOLOGY OF A LIE GROUP

We now have all the ingredients for our proof. Consider the map $\psi: G/T \times T \rightarrow G$ given by $\psi(gT, t) = gtg^{-1}$. The Weyl group W acts on T by conjugation and on G/T by $w \cdot gT = gn^{-1}T$, where $w = nT$. Hence W acts on $H(G/T \times T) = H(G/T) \otimes H(T)$. Since $\psi(gn^{-1}T, wtw^{-1}) = \psi(gT, t)$, it follows that the induced map ψ^* on cohomology maps $H(G)$ to $[H(G/T) \otimes H(T)]^W$. Though we prefer to have it in this form, the latter group could be thought of as the cohomology of the quotient of $G/T \times T$

by the action of W , and this quotient has a natural interpretation. As in the introduction, let M be the set of pairs (g, T') where T' is a maximal torus in G containing $g \in G$. The map $G/T \times_W T \rightarrow M$ sending $(gT, t) \bmod(W)$ to (gtg^{-1}, gTg^{-1}) is a diffeomorphism.

PROPOSITION (6.1). *The map induced by ψ on cohomology is an isomorphism of graded rings*

$$\psi^*: H(G) \xrightarrow{\cong} [H(G/T) \otimes H(T)]^W.$$

Proof. We compute the derivative $(d\psi)_{(gT, t)}$ at a point $(gT, t) \in G/T \times T$. For each point $gT \in G/T$, we identify \mathfrak{m} with the tangent space $T_{gT}(G/T)$ by letting $X \in \mathfrak{m}$ correspond to the initial tangent vector X_{gT} of the path $s \mapsto g(\exp sX)T$ in G/T . Similarly, an element $X \in \mathfrak{g}$ (resp. $H \in \mathfrak{t}$) corresponds to a tangent vector $X_g \in T_g(G)$ (resp. $H_t \in T_t(T)$, for $t \in T$).

Then

$$\begin{aligned} (d\psi)_{gT, t}(X_{gT}, 0) &= \frac{d}{ds} g(\exp sX) t(\exp -sX) g^{-1} \Big|_{s=0} \\ &= \frac{d}{ds} gtg^{-1} [\exp s \operatorname{Ad}(gt^{-1})X] [\exp -s \operatorname{Ad}(g)X] \Big|_{s=0} \\ &= \frac{d}{ds} gtg^{-1} [X + s \operatorname{Ad}(g) (\operatorname{Ad}(t^{-1}) - 1)X + O(s^2)] \Big|_{s=0} \\ &= [\operatorname{Ad}(g) (\operatorname{Ad}(t^{-1}) - 1)X]_{gtg^{-1}}. \end{aligned}$$

Similarly, we find, for $H \in \mathfrak{t}$, that

$$(d\psi)_{gT, t}(0, H_t) = [\operatorname{Ad}(g)H]_{gtg^{-1}}.$$

Hence, under the identifications, $(d\psi)_{(gT, t)}$ is the map

$$(\operatorname{Ad}(t^{-1}) - I)_{\mathfrak{m}} \oplus I: \mathfrak{m} \oplus \mathfrak{t} \rightarrow \mathfrak{m} \oplus \mathfrak{t} = \mathfrak{g}.$$

Here the subscript \mathfrak{m} means we view $\operatorname{Ad}(t^{-1}) - I$ as a map from \mathfrak{m} to itself. Now G being compact and connected, we must have $\det \operatorname{Ad}(t) = 1$, so

$$\det (d\psi)_{(gT, t)} = \det (I - \operatorname{Ad}(t))_{\mathfrak{m}}.$$

(Actually, \mathfrak{m} is always even-dimensional as we have seen, so there is no need to reverse the subtraction).

We compute the degree of ψ by finding a regular value. Let t_0 be a generic element in T , as in (2.3). Consider $\psi^{-1}(t_0) = \{(gT, t): gtg^{-1} = t_0\}$. It turns out that any two elements of T conjugate in G must be conjugate by an element

of W . (In $U(n)$, two diagonal matrices with the same set of eigenvalues must be conjugate by a permutation matrix.) It follows easily then that

$$\psi^{-1}(t_0) = \{(wT, wt_0 w^{-1}) : w \in W\}.$$

We next show that ψ preserves orientation at each point in $\psi^{-1}(t_0)$. The eigenvalues of $Ad(t_0)$ in \mathfrak{m} are complex conjugate pairs z, \bar{z} , where $|z| = 1, z \neq 1$. Hence $|1 - z||1 - \bar{z}| = 2(1 - \operatorname{Re}(z)) > 0$, so $\det(I - Ad(t_0))_{\mathfrak{m}} > 0$.

At this point we know the degree of ψ is $\deg \psi = |W| \neq 0$. By Poincaré duality, any smooth map between compact manifolds of the same dimension is injective on cohomology as soon as it has nonzero degree. Hence $\psi^*: H(G) \rightarrow [H(G/T) \times H(T)]^W$ is injective. We finish the proof of (6.1) by showing that both sides have the same dimension.

For this we use, three times, the following basic principle. Let K be a compact group (here K will be G, T or W). Let dk be the left invariant Haar measure on K with total mass one. Let V be a finite dimensional real vector space, and $\rho: K \rightarrow GL(V)$ a continuous group homomorphism. Then the space V^K of vectors fixed by all $\rho(k), k \in K$, has dimension

$$\dim V^K = \int_K \operatorname{trace} \rho(k) dk.$$

To compute this integral over G , we must exploit further the computation of $d\psi$. Let $\omega_G, \omega_T, \omega_{G/T}$ be the unique invariant (under left translations by G, T , and G respectively) differential forms of top degree whose integral over the respective manifold is one. The pull-back formula for integration gives

$$\int_G f \omega_G = \frac{1}{\deg \psi} \int_{G/T \times T} f \circ \psi(gT, t) |\det(d\psi)_{(gT, t)}| \omega_{G/T} \wedge \omega_T,$$

where the determinant is computed with respect to bases spanning parallelograms of unit volume with respect to the appropriate forms. Taking f to be invariant under conjugation by G , we arrive at the famous Weyl integration formula:

$$\int_G f \omega_G = \frac{1}{|W|} \int_T f(t) \det(I - Ad(t))_{\mathfrak{m}} \omega_T.$$

Expand the function $t \mapsto \det(I - Ad(t))_{\mathfrak{m}}$ in a sum of characters of T : $n_0 \chi_0 + n_1 \chi_1 + \cdots + n_k \chi_k$. Here χ_0 is the trivial character of T ,

appearing n_0 times, and for $i > 0$ each χ_i is a nontrivial character appearing n_i times. Taking for f the constant function equal to one, and applying the basic principle of invariants to T , we find $n_0 = |W|$.

Taking for f the function $f(g) = \det(I + Ad(g))$, i.e., the trace of $Ad(g)$ acting on $\Lambda \mathfrak{g}$, we find, using the Cartan-de Rham isomorphism (4.3), that

$$\begin{aligned} \dim H(G) &= \dim(\Lambda \mathfrak{g})^G = \int_G \det(I + Ad(g)) \omega_G \\ &= \frac{1}{|W|} \int_T \det(I + Ad(t)) \det(I - Ad(t))_{\mathfrak{m}} \omega_T \\ &= \frac{2^{\dim T}}{|W|} \int_T \det(I - Ad(t^2))_{\mathfrak{m}} \omega_T. \end{aligned}$$

Now the squaring map on T is surjective, so the square of a nontrivial character of T is still nontrivial. Hence the trivial character again appears with multiplicity $|W|$ in the expansion of $\det(I - Ad(t^2))_{\mathfrak{m}}$. This multiplicity is the value of the integral, so $\dim H(G) = 2^{\dim T} = 2^l$.

On the other hand, we saw in (5.3) that the trace of $w \in W$ acting on $H(G/T)$ is $|W|$ if $w = 1$, zero otherwise. Applying the invariance formula one more time, we find that $\dim [H(G/T) \otimes H(T)]^W = 2^l$ as well, completing the proof of (6.1). \square

We now have the main result

(6.2) THEOREM. *The cohomology ring $H(G)$ with real coefficients is a bigraded exterior algebra with generators in bi-degrees $(2m_i, 1)$, for $1 \leq i \leq l$.*

Proof. By (6.1) and (5.4), we have

$$H(G) \simeq [H(G/T) \otimes H(T)]^W \simeq [\mathcal{H}_{(2)} \otimes \Lambda]^W,$$

and by (3.8), the latter space is an exterior algebra with generators in degrees $(2m_i, 1)$, for $1 \leq i \leq l$. \square

Moreover, from the multiplicity formula (3.8), the dimensions of the bi-graded pieces are given in terms of the exponents as follows

(6.3) COROLLARY. *For each $q \geq 0$, we have*

$$\sum_{n=0}^{\dim G} \dim [H^{n-q}(G/T) \otimes H^q(T)]^W u^n = u^q s_q(u^{2m_1}, \dots, u^{2m_l}).$$

(6.4) We give two interpretations of the bigrading. First, we follow [L] and consider the spectral sequence of the fibration $G \rightarrow G/T$, which has E_2 -term

$$E_2^{pq} = H^p(G/T) \otimes H^q(T),$$

and converges to $H(G)$. This spectral sequence does not degenerate at E_2 , but it has a spectral subsequence which does degenerate, and still converges to $H(G)$.

To see this we again consider the Weyl group action. More precisely, N acts by automorphisms of the fibration $G \rightarrow G/T$, which in turn induce automorphisms of each term in the spectral sequence, commuting with the differentials. On $E_2^{pq} = H^p(G/T) \otimes H^q(T)$, the action of N factors through W and is the same as that considered above. Thus we have representations of W on the spaces E_2^{pq} , hence on each E_r^{pq} for $r \geq 2$.

For each p, q, r we decompose $E_r^{pq} = (E_r^{pq})^W \oplus (E_r^{pq})_W$, where the subscript W indicates a W -stable complement to the invariants. Each of the latter two spaces is a spectral subsequence, and since E_∞^{pq} is a subquotient of $H^{p+q}(G)$ and N acts trivially on $H(G)$ (because G is connected), we must have $(E_\infty^{pq})_W = 0$. On the other hand, $(E_\infty^{pq})^W$ is a subquotient of $(E_2^{pq})^W = [H^p(G/T) \otimes H^q(T)]^W$, so we have

$$\begin{aligned} 2^l = \dim H(G) &= \sum_{p,q} \dim (E_\infty^{pq})^W \leq \sum_{p,q} \dim (E_2^{pq})^W \\ &= \sum_q \dim [H(G/T) \otimes \Lambda^q]^W = 2^l. \end{aligned}$$

It follows that $\dim (E_\infty^{pq})^W = \dim (E_2^{pq})^W$ for all p, q , so the spectral subsequence of W -invariants degenerates at $(E_2)^W$, and (6.1) is proved again.

(6.5) The significance of the bigrading on $H(G)$ can be seen in yet another way, inspired by [GHV]. We consider, for a fixed integer $k \neq 1$, the k^{th} -power maps $x \mapsto x^k$, denoted p_k and P_k , on T and G , respectively. It is shown in [GHV] that the Lefschetz number of P_k equals that of p_k , namely $(1-k)^l$. Let $H^n(G)_q$ be the k^q -eigenspace of P_k^* acting on $H^n(G)$. It is further shown in [GHV] that $\sum_n \dim H^n(G)_q = \binom{l}{q}$. We can refine this by computing each $\dim H^n(G)_q$ separately. Consider the commutative diagram

$$\begin{array}{ccc}
 H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W. \\
 P_k^* \downarrow & & \downarrow 1 \otimes p_k^* \\
 H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W.
 \end{array}$$

Since p_k^* acts by k^q on $H^q(T)$, (6.1) implies that $H^n(G)_q \simeq [H^{n-q}(G/T) \otimes H^q(T)]^W$, and (6.3) gives the dimension of the latter space.

(6.6) This last interpretation of the bigrading shows that it is natural in the following sense. Suppose $f: K \rightarrow G$ is a homomorphism between two compact connected Lie groups. Since f commutes with the power maps P_k on G and K , the cohomology map f^* sends $H^n(G)_q$ to $H^n(K)_q$. Suppose for example that K is a closed connected subgroup of G and f is the inclusion map. Choose, as we may, a maximal torus T of G such that $S := T \cap K$ is a maximal torus of K . The restriction map $H(G) \rightarrow H(K)$ becomes, via (6.1), the map $[H(G/T) \otimes H(T)]^W \rightarrow [H(K/S) \otimes H(S)]^{W_K}$ induced by restriction on each factor, where W_K is the Weyl group of S in K .

(6.7) We close with the homology interpretation of (6.1), which says the homology map ψ_* induced by ψ is surjective. It follows that the homology of G is spanned by the cycles $[\psi(\bar{X}_w, T_I)] = \{gtg^{-1}: gT \in \bar{X}_w, t \in T_I\}$. Here $w \in W$, X_w is the Schubert cell (see (5.2)) and $T_I = \prod_{i \in I} T_i$, where $T = T_1 \times \cdots \times T_l$, with each $T_i \simeq S^1$. Using the results in [BGG], one can explicitly write down the action of W on $H_*(G/T)$ in terms of the Schubert cell basis, and this leads, in principle, to the linear relations in $H_*(G)$ satisfied by the cycles $[\psi(\bar{X}_w, T_I)]$.

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