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(4.2) Assume that K is connected. Since any homomorphism from K to the multiplicative real numbers must be trivial, the determinant is a nonzero element of the one dimensional space $(\Lambda^n \mathfrak{n}^*)^K$, where $n = \dim M$. It follows that M is orientable, and any G-invariant n-form on M will have nonzero integral over M as soon as it does not vanish at one point.

(4.3) In the case $M = G$ we have the additional symmetry of left and right multiplication by $G \times G$, and every cohomology class contains a bi-invariant representive. The value at e of a bi-invariant form is $Ad(G)$ invariant. Taking the derivative of the condition for $\omega \in \Lambda^p \mathfrak{g}^*$ to be $Ad(G)$-invariant, we find (product rule) that $\omega([X, X_1], X_2, \ldots, X_p) + \cdots + \omega(X_1, \ldots, [X, X_p]) = 0$ for all $X, X_1, \ldots, X_p \in \mathfrak{g}$. It is then not hard to show that this condition implies that $\delta\omega = 0$. Hence all bi-invariant forms are closed. Since δ commutes with Ad, it follows that the de Rham cohomology of G is computed by the complex $(\Lambda \mathfrak{g}^*)^G$, with zero differential. That is, $H(G) \simeq (\Lambda \mathfrak{g}^*)^G$, as graded rings.

5. THE COHOMOLOGY OF FLAG MANIFOLDS

The *Bruhat Decomposition* is a cell decomposition of the flag manifold G/T into even dimensional cells indexed by elements of the Weyl group W. It generalizes the decomposition of the two-sphere (flag manifold of $SU(2)$) into a point and an open disk. The existence of such a decomposition implies that there are no boundary maps in cellular homology, and the cohomology of $H(G/T)$ is nonzero only in even degrees.

It is customary to explain the The Bruhat decomposition in terms of complex groups. For example the flag manifold for $U(n)$ is in fact a homogeneous space for $GL_n(\mathbf{C})$, and the cells can be described as the orbits of certain subgroups of the group of upper triangular complex matrices, which do not live in $U(n)$. We shall, however, describe the cell decomposition of G/T purely in terms of the compact group G, using Morse theory. It was Bott, later with Samelson, who first applied Morse theory to the loop space of G from which, combined with results of Borel and Leray, they deduced results on the topology of G and G/T. See [BT] for a brief introduction to Morse theory.

(5.1) We need to find a "Morse function" f on G/T. This is a smooth real valued function whose Hessian (matrix of second partial derivatives taken in local coordinates) at each critical point has nonzero determinant. How shall we find one? For the unit sphere in $\mathbf{R}^3$ centered at $(0, 0, 0)$, we can take

$f(x, y, z) = z$, and the critical points are the north and south poles, where the Hessian is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. The flow lines of the gradient of f emanating from the south pole form a 2-cell, and the north pole is a zero-cell. We can also write f using the dot product on $\mathbf{R}^3$ as $f(p) = p \cdot n$, where n is the position vector of the north pole. Viewing $\mathbf{R}^3$ as the Lie algebra $\mathfrak{su}(2)$, this tells us what to do in general.

As analogue of the north pole, we take $H_0 \in \mathfrak{t}$ to be the regular element defining the positive roots, as in (2.3). Since the $Ad(G)$-centralizer of H_0 is exactly T, we may view $G/T \subset \mathfrak{g}$ as the $Ad(G)$-orbit of H_0 (analogous to $S^2 \subset \mathbf{R}^3$), and we define a function $f: G/T \to \mathbf{R}$ by

$$f(gT) = \langle Ad(g)H_0, H_0 \rangle \; .$$

For $X \in \mathfrak{g}$, let $\bar{X}$ be the vector field on G/T given, for a smooth function ϕ on G/T, by

$$\bar{X}\phi(gT) = \frac{d}{ds}\, \phi((\exp sX)\, gT)\,|_{s=0} \; .$$

Then a short computation, using the ad-invariance of the inner product, shows that

$$\bar{X}f(gT) = \langle Ad(g)H_0, [H_0, X] \rangle \; .$$

Since the centralizer of H_0 in $\mathfrak{g}$ is exactly $\mathfrak{t}$, the image of $ad(H_0)$ is all of $\mathfrak{m}$. So gT is a critical point of f if and only if $\langle Ad(G)H_0, \mathfrak{m} \rangle = 0$, forcing $Ad(g)H_0 \in \mathfrak{t}$. It then follows that $Ad(g)H_0 = Ad(w)H_0$ for some $w \in W$. So the critical points of f are the wT, for $w \in W$.

Let $X_1, X_2, \ldots, X_{2\nu}$ be the orthonormal basis of $\mathfrak{m}$ from (2.3). For each $w \in W$, the differential of the projection $\pi: G \to G/T$ maps $Ad(w)\mathfrak{m} = \mathfrak{m}$ isomorphically onto $T_{wT}(G/T)$, so we can use the X_i's to compute the Hessian of f at each point wT. Let $h_{ij}(w)$ be the ij entry in the Hessian matrix. Another short computation gives

$$h_{ij}(w) = \bar{X}_i \bar{X}_j f(wT) = \langle [X_i, Ad(w)H_0], [H_0, X_j] \rangle \; .$$

Recalling the bracket relations in (2.3), we find that

$$h_{ii}(w) = -\alpha_i(Ad(w)H_0)\,\alpha_i(H_0) \; ,$$

and $h_{ij}(w) = 0$ if $i \neq j$. The regularity of H_0 implies that of $Ad(w)H_0$, so the Hessian is nonsingular. It follows that the index of the critical point wT, by definition the number of negative eigenvalues of the Hessian, is twice the number $m(w)$ of positive roots α such that $w^{-1}\alpha$ is again positive. Now by the main theorem of Morse theory, the Poincaré polynomial of G/T is $\sum_{w \in W} u^{2m(w)}$. In particular, $H^{\text{odd}}(G/T) = 0$ and $\dim H^{\text{even}}(G/T) = \dim H(G/T) = \chi(G/T) = |W|$.

(5.2) The *Schubert cell* X_w in the Bruhat decomposition is spanned by those flow lines of the gradient of f which emanate from wT. The dimension of this cell then equals twice the number of positive eigenvalues of the Hessian at wT, which is the number of positive roots made negative by w.

(5.3) Recalling that W acts on G/T, we use Leray's argument to determine the structure of the W-module $H(G/T)$, ignoring the grading for now. The element $w \in W$ acts by $w \cdot (gT) = gw^{-1}T$ of W on G/T. Since there is no cohomology in odd degrees, the Lefschetz number of w equals its trace on $H(G/T)$. If $w \neq 1$ there are no fixed points so the Lefschetz number is zero. If $w = 1$ we are computing the Euler characteristic which we now know is $|W|$. Hence the trace of any $w \in W$ acting on $H(G/T)$ is that of the regular representation, so $H(G/T) \simeq \mathbf{R}[W]$ (the group ring of W) as W-modules.

The theorem of Borel is a refinement of this, and describes the W-module structure of $H(G/T)$ in each degree. Recall the graded ring $\mathscr{S}$ of polynomial functions on $\mathfrak{t}$ and its ideal $\mathscr{I}$ generated by the W-invariant polynomials of positive degree. Our object is to prove the following

(5.4) THEOREM (Borel). *There is a degree-doubling W-equivariant ring isomorphism*

$$c\colon \mathscr{S}/\mathscr{I} \to H(G/T)\,.$$

Consequently, $\mathscr{H}_{(2)} \simeq H(G/T)$, where $\mathscr{H}_{(2)}$ is $\mathscr{H}$ with the grading degrees doubled.

Proof. We will describe the cohomology ring of G/T in terms of G-invariant differential forms. For each $\lambda \in \mathfrak{t}^*$, extended to a functional on all of $\mathfrak{g}$ by making it zero on $\mathfrak{m}$, define an $Ad(T)$-invariant two-form ω_λ on $\mathfrak{m}$ by

$$\omega_\lambda(X, Y) = \lambda([X, Y])\,.$$

As in (4.1), this corresponds to a left-invariant form $\tilde{\omega}_\lambda$ on G/T.

Though it is not needed here, one can show that if λ is the differential of a character $\chi_\lambda\colon T \to S_1$, then $\frac{1}{4\pi}\omega_\lambda$ represents the first Chern class of the corresponding complex line bundle $G \times_T \mathbf{C}$, where T acts on $\mathbf{C}$ via χ_λ.

Returning to the proof, note that for $w \in W$, acting on $\mathfrak{t}^*$ by $w\lambda(H) = \lambda(Ad(w)^{-1}H)$, and on the space of differential forms $\Omega(G/T)$ via its action on G/T, we have $w^*\omega_\lambda = \omega_{w\lambda}$. Moreover, the Jacobi identity says that $\delta\omega_\lambda = 0$, and we let $c(\lambda) = [\tilde{\omega}_\lambda] \in H^2(G/T)$ be the cohomology class of $\tilde{\omega}_\lambda$. This extends to a degree-doubling ring homomorphism

$$c: \mathscr{S} \to H(G/T)$$

which also preserves the W-action on both sides. Since $H(G/T)$ is the regular representation of W, its W-invariants are one-dimensional and therefore can occur only in $H^0(G/T)$. By the W-equivariance, it follows that the kernel of c contains the ideal $\mathscr{I} \in \mathscr{S}$ generated by W-invariant polynomials of positive degree. Borel's theorem asserts that $\mathscr{I}$ is exactly the kernel of c.

Since $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$ (see (3.4)), we shall prove the theorem by showing that the restriction of c to $\mathscr{H}$ is injective. We start in the top dimension, where our task is to show that $c(\Pi)$ (recall from (3.5) that Π is the primordial harmonic polynomial) is nonzero in $H^{2\nu}(G/T)$. One way is to use the Chern class interpretation to show that $c(\Pi)$ is a nonzero multiple of the Euler class of G/T, whose integral over G/T is $\chi(G/T) = |W| \neq 0$. However, we shall be more pedestrian about it, and evaluate $c(\Pi)$ on a basis on $\mathfrak{m}$ (see (4.2)).

Recall that for each positive root α_i, we have elements $X_i, X_{i+\nu}$ in $\mathfrak{m}$, with bracket relations $[X_i, X_{i+\nu}] = H_i \in \mathfrak{t}$, $[X_i, X_j] \in \mathfrak{m}$ if $j \neq i+\nu$. The set $\{X_i\colon 1 \leqslant i \leqslant 2\nu\}$ is a basis of $\mathfrak{m}$. Write ω_i for ω_{α_i}, so $c(\Pi) = [\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_\nu]$. We compute

$$\begin{aligned} &\omega_1 \wedge \cdots \wedge \omega_\nu(X_1, X_{1+\nu} \ldots, X_\nu, X_{2\nu}) \\ &= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} sgn(\sigma)\omega_1(X_{\sigma(1)}, X_{\sigma(1+\nu)}) \cdots \omega_\nu(X_{\sigma(\nu)}, X_{\sigma(2\nu)}) \\ &= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} sgn(\sigma)\alpha_1([X_{\sigma(1)}, X_{\sigma(1+\nu)}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\sigma(2\nu)}]) \,. \end{aligned}$$

Now $\alpha_i([X_{\sigma(i)}, X_{\sigma(i+\nu)}]) = 0$ unless $[X_{\sigma(i)}, X_{\sigma(i+\nu)}] \in \mathfrak{t}$, so the σ^{th} term is nonzero only if σ permutes the pairs $\pi_i = \{i, i+\nu\}$ and possibly switches some of the members of each pair. Moreover, $sgn(\sigma)$ equals minus one to the number of switches, so we get

$$\begin{aligned} &\omega_1 \wedge \cdots \wedge \omega_\nu(X_1, X_{1+\nu} \ldots, X_\nu, X_{2\nu}) \\ &= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1([X_{\sigma(1)}, X_{\nu+\sigma(1)}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\nu+\sigma(\nu)}]) \\ &= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1(H_{\sigma(1)}) \cdots \alpha_\nu(H_{\sigma(\nu)}) \\ &= \frac{2^\nu}{(2\nu)!} \partial_1 \cdots \partial_\nu \Pi \,, \end{aligned}$$

where, as in (3.3), ∂_i is the derivation of $\mathscr{S}$ extending the functional $\lambda \mapsto \lambda(H_i)$. We have a perfect pairing

$$\mathscr{D} \otimes \mathscr{S} \to \mathbf{R}$$

given by $(D, f) = (Df)(0)$. Since the pairing is perfect, something in degree ν must pair nontrivially with Π. Since an irreducible W-module can only pair nontrivially with its dual, and the self-dual character ε occurs with multiplicity one in $\mathscr{D}^\nu$, afforded by $\partial_1 \cdots \partial_\nu$, we must have $\partial_1 \cdots \partial_\nu \Pi \neq 0$, so $c(\Pi) \neq 0$.

Observe that $\partial_1 \cdots \partial_\nu$ is analogous to the fundamental class of G/T, and the pairing is essentially that between homology and cohomology. We further remark that in fact all irreducible representations of W are defined over the rational numbers, hence they are all self dual. This is a consequence of Springer's cohomological construction of W-modules [Sp].

Returning again to our task, we now inductively assume that $c: \mathscr{H}^k \to H^{2k}(G/T)$ is injective for $k \leqslant \nu$, and let $V = \mathscr{H}^{k-1} \cap \ker c$. Note that V is preserved by W since c is W-equivariant. The sign character does not occur in $\mathscr{H}^{k-1}$, so there is a positive root α whose corresponding reflection s_α does not act by $-I$ on V. Decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α. If $V \neq 0$ then $V_+ \neq 0$, so take $f \in V_+$. Now $c(\alpha f) = c(\alpha) c(f) = 0$, and αf is in degree k, so we must have $\alpha f \in \mathscr{I}$ by the induction hypothesis. Let $h_1, \ldots, h_{|W|}$ be a basis of $\mathscr{H}$ with $h_1, \ldots, h_r$ s_α-skew and the rest s_α invariant. By Chevalley's theorem (3.2), we can write $\alpha f = \sum h_i \sigma_i$ with σ_i W-invariant of positive degree. Since αf is s_α-skew, the sum only goes up to r. Now for $i \leqslant r$, the polynomial h_i must vanish on $\ker \alpha$, hence can be written $h_i = \alpha h_i'$ for some $h_i' \in \mathscr{S}$. But then $f = \sum_{i=1}^r h_i' \sigma_i \in \mathscr{I}$. Since f is supposed to be harmonic, we must have $f = 0$. Hence c is injective on $\mathscr{H}$, and the proof of Borel's theorem is complete. $\square$

6. The cohomology of a Lie group

We now have all the ingredients for our proof. Consider the map $\psi: G/T \times T \to G$ given by $\psi(gT, t) = gtg^{-1}$. The Weyl group W acts on T by conjugation and on G/T by $w \cdot gT = gn^{-1}T$, where $w = nT$. Hence W acts on $H(G/T \times T) = H(G/T) \otimes H(T)$. Since $\psi(gn^{-1}T, wtw^{-1}) = \psi(gT, t)$, it follows that the induced map ψ^* on cohomology maps $H(G)$ to $[H(G/T) \otimes H(T)]^W$. Though we prefer to have it in this form, the latter group could be thought of as the cohomology of the quotient of $G/T \times T$