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Autor: Reeder, Mark

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184 M. REEDER

element  $H_0 \in \mathfrak{t}$ . We may and shall choose the positive roots so that they take strictly positive values on  $H_0$ . The action of W on  $\mathfrak{t}$  is generated by reflections about the kernels of the positive roots.

Since each  $\mathfrak{m}_i$  is also preserved by  $ad(\mathfrak{t})$ , we can choose an orthonormal basis  $\{X_i, X_{v+i}\}$  of  $\mathfrak{m}_i$  such that, for  $H \in \mathfrak{t}$ , the matrix of  $ad(H)|_{\mathfrak{m}_i}$  with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the *ad*-invariance of the inner product  $\langle , \rangle$  implies, for all  $1 \leq i \leq v$ , all  $1 \leq j \leq 2v$  and all  $H \in \mathfrak{t}$  that

$$\langle H, [X_i, X_j] \rangle = \langle [H, X_i], X_i \rangle = -\alpha_i(H) \langle X_{i+v}, X_i \rangle$$
.

By orthonormality, this last pairing can only be nontrivial if j = i + v. Hence if  $j \neq i + v$ , we have  $[X_i, X_j] \in \mathfrak{m}$ . The same thing happens if i > v and  $j \neq i - v$ .

On the other hand, for  $1 \le i \le v$ , set  $H_i = [X_i, X_{v+i}]$ . This is Ad(T)-invariant, so  $H_i \in \mathfrak{t}$ , and  $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$ . It follows that the span of  $X_i, X_{i+v}, H_i$  is a Lie subalgebra  $\mathfrak{g}_i$  of  $\mathfrak{g}$ . It is always isomorphic to  $\mathfrak{su}(2)$ .

## 3. Invariant Theory

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

# (3.1) Let

$$\mathcal{S} = \bigoplus_{p=0}^{\infty} \mathcal{S}^p$$
 and  $\Lambda = \bigoplus_{q=0}^{l} (l = \dim \mathfrak{t})$ 

be the symmetric and exterior algebras on  $t^*$ , respectively. The adjoint action of W on t induces representations of W on S and  $\Lambda$  by degree-preserving algebra automorphisms. For example, the action of W on  $\Lambda^l$  is multiplication by the  $sign\ character$ 

$$\varepsilon: W \to \{\pm 1\}$$
 given by  $\varepsilon(w) = \det Ad(w)_{t}$ .

Note that  $\varepsilon(w)$  is the parity of the number of reflections needed to express  $Ad(w)_{t}$ .

We are interested in W-invariant polynomials, and more generally, W-invariant differential forms with polynomial coefficients. For the unitary group U(n), the ring of invariants  $\mathcal{S}^W$  is generated by the elementary symmetric polynomials  $s_1, ..., s_n$  in variables  $x_1, ..., x_n$  defined as

$$s_d(x_1, ..., x_n) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}.$$

The elementary symmetric polynomials are algebraically independent, and their number equals the dimension n of a maximal torus of U(n). In general, we have

(3.2) Theorem (Chevalley). The ring  $\mathscr{S}^W$  has algebraically independent homogeneous generators  $F_1, ..., F_l$ , hence is a polynomial ring

$$\mathcal{S}^W = \mathbf{R}[F_1, ..., F_l] .$$

We number these generators so that  $\deg F_1 \leqslant \deg F_2 \leqslant \cdots \leqslant \deg F_l$ . (Note to experts: Since we are not assuming G to be semisimple, some of the  $F_i$ 's could have degree one.) The exponents  $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_l$  of W acting on t are defined by the relations  $m_i + 1 = \deg F_i$ . It is known that  $m_1 + \cdots + m_l = v$ , and  $(1 + m_1) \cdots (1 + m_l) = |W|$ .

Every compact connected Lie group is, up to finite covering, the product of a central torus with a direct product of classical groups SU(n), SO(n), Sp(n), and exceptional groups  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ ,  $E_8$ . For these groups the  $m_i$ 's are given as follows:

$$SU(n)$$
: 1, 2, ...,  $n-1$ .  $SO(2n)$ : 1, 3, ...,  $2n-3$ ,  $n-1$ .  $SO(2n+1)$  and  $Sp(n)$ : 1, 3, ...,  $2n-1$ .  $G_2$ : 1, 5.  $F_4$ : 1, 5, 7, 11.  $E_6$ : 1, 4, 5, 7, 8, 11.  $E_7$ : 1, 5, 7, 9, 11, 13, 17.  $E_8$ : 1, 7, 11, 13, 17, 19, 23, 29.

These numbers are easy to verify for the classical groups and  $G_2$  (whose maximal torus T is that of SU(3) with Weyl group  $S_3$  extended by the inverse map on T), using elementary symmetric polynomials as above. Computing the exponents for the other exceptional groups is more difficult. See [C].

(3.3) The W-module structure of the whole polynomial ring  $\mathscr S$  is given as follows. Let  $\mathscr D$  be the ring of constant coefficient differential operators on  $\mathscr S$ . We can think of  $\mathscr D$  as the symmetric algebra  $S(\mathfrak t)$ , where  $H \in \mathfrak t$ 

186 M. REEDER

corresponds to the derivation of  $\mathscr{S}$  extending the functional on  $\mathfrak{t}^*$  given by evaluation at H. Then W acts naturally on  $\mathscr{D}$  and one defines the "harmonic polynomials" in  $\mathscr{S}$  to be those annihilated by the W-invariant differential operators:

$$\mathcal{H} = \{ f \in \mathcal{S} \colon \mathcal{D}^W f = 0 \} \ .$$

Let  $\mathcal{H}^p = \mathcal{H} \cap \mathcal{S}^p$ . Then  $\mathcal{H} = \bigoplus_p \mathcal{H}^p$ , since a differential operator is W-invariant only if each of its homogeneous components is so. The action of W on  $\mathcal{S}$  leaves  $\mathcal{H}$  invariant.

Let  $\mathscr{I}$  be the ideal in  $\mathscr{S}$  generated by the elements of  $\mathscr{S}^W$  of positive degree. It is known (see [H, p. 360] that  $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$ , and the multiplication map is a linear isomorphism  $\mathscr{H} \otimes \mathscr{S}^W \xrightarrow{\sim} \mathscr{S}$ . The former implies that  $\mathscr{S}/\mathscr{I}$  and  $\mathscr{H}$  are isomorphic W-modules. They are in fact isomorphic to the regular representation of W, as we shall see in (5.4). The isomorphism  $\mathscr{H} \otimes \mathscr{S}^W \simeq \mathscr{S}$  implies the identity

$$\sum_{p \geq 0} \dim \mathcal{H}^{p} t^{p} = \prod_{i=1}^{l} (1 + t + t^{2} + \cdots + t^{m_{i}}),$$

which in turn shows that dim  $\mathcal{H}^{\nu} = 1$ , and  $\mathcal{H}^{p} = 0$  for  $p > \nu$ .

(3.4) Let V be any irreducible W-module. Suppose V is a constituent of  $\mathcal{S}^b$ , and not a constituent of  $\mathcal{S}^c$ , for any c < b. We call b the birthday of V. Then the V-isotypic component of  $\mathcal{S}^b$  must consist of harmonic polynomials, for otherwise, a W-invariant differential operator of positive degree would intertwine V with a space of polynomials of lower degree.

For example, the primordial harmonic polynomial is

$$\Pi = \prod_{\alpha \in \Delta^+} \alpha \in \mathcal{H}^{\vee},$$

where we recall that  $\Delta^+$  is the set of positive roots. For U(n),  $\Pi$  is the van der Monde determinant  $\prod_{i < j} x_i - x_j$ , which transforms under the symmetric group  $S_n$  by the sign character. In general,  $\Pi$  transforms by the sign character  $\varepsilon$  of W, and any other polynomial transforming by  $\varepsilon$  must vanish on all root hyperplanes, hence be divisible by  $\Pi$ . Therefore  $\Pi$  is harmonic,  $\nu$  is the birthday of  $\varepsilon$  and (1.4) shows that  $\mathcal{H}^{\nu}$  is spanned by  $\Pi$ .

We say that  $\Pi$  is primordial because  $\mathscr{H}$  is spanned by the partial derivatives of  $\Pi$  (see [S]). This turns out to be the algebraic analogue of Poincaré duality for G/T.

As we have seen, the sign character is also afforded by  $\Lambda^{I}$ . In general, if g is simple then each exterior power  $\Lambda^{q}$  is an irreducible W-module. We shall determine the birthday of each  $\Lambda^{q}$  shortly.

- (3.5) Now consider the algebra  $\mathscr{S} \otimes \Lambda$  of differential forms on t with polynomial coefficients. Let  $F_1, ..., F_l$  be homogeneous generators of  $\mathscr{S}^W$  as in (3.2). Extending that result, Solomon [Sol] has described the W-invariants in  $\mathscr{S} \otimes \Lambda$ . Because it seems not so well known but is important here, we give a proof, taken from [H].
- (3.6) Theorem (Solomon). The space  $(\mathscr{S} \otimes \Lambda)^W$  of W-invariants in  $\mathscr{S} \otimes \Lambda$  is a free  $\mathscr{S}^W$ -module with basis

$$\{dF_{i_1} \wedge \cdots \wedge dF_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant l\}$$
.

*Proof.* It is a general fact about polynomials that the algebraic independence of  $F_1, ..., F_l$  is equivalent to the form  $dF_1 \wedge \cdots \wedge dF_l$  not being identically zero. Let  $x_1, ..., x_l$  be a basis of  $t^*$ . Then

$$dF_1 \wedge \cdots \wedge dF_l = Jdx_1 \cdots dx_l$$
,

where the Jacobian J is a polynomial of degree  $m_1 + \cdots + m_l = v$ . The left side is W-invariant and  $dx_1 \wedge \cdots \wedge dx_l$  affords the sign character  $\varepsilon$ . Hence J must also afford  $\varepsilon$  and, because of its degree, J must be a nonzero multiple of the primordial harmonic polynomial  $\Pi$ . Thus

$$dF_1 \wedge \cdots \wedge dF_l = c \prod dx_1 \wedge \cdots \wedge dx_l$$

for some nonzero real number c.

For a sequence  $I=i_1<\cdots< i_q$ , let I' be the increasing sequence of all integers in  $\{1,\ldots,l\}-\{i_1,\ldots,i_q\}$ . Set  $dF_I=dF_{i_1}\wedge\cdots\wedge dF_{i_q}$  for any sequence I. Let k be the quotient field of  $\mathscr{G}$ . If  $f_I\in k$  are such that  $\sum_I f_I dF_I=0$  then multiplying by  $dF_{I'}$  kills all terms but I, leaving  $\pm cf_I \Pi dx_1\cdots dx_l=0$ , so  $f_I=0$ . Counting dimensions, we find that the  $dF_I$  are a k-basis of  $k\otimes \Lambda$ , and are in particular linearly independent over  $\mathscr{S}^W$ . Now suppose  $\omega\in\mathscr{S}\otimes\Lambda$  is W-invariant. We can express  $\omega=\sum_I g_I dF_I$  for some  $g_I\in k$ . Multiplying by  $dF_{I'}$  again, we have

$$\omega \wedge dF_{I'} = \pm cg_I \Pi dx_1 \cdots dx_I \in [\mathscr{S} \otimes \Lambda]^W$$
.

This forces  $g_I$  to be not only W-invariant, but also polynomial.

For  $\omega \in \mathcal{S} \otimes \Lambda$ , let  $\omega' \in \mathcal{S}/\mathcal{I} \otimes \Lambda$  be obtained by reducing the coefficients of  $\omega$  modulo  $\mathcal{I}$ . This induces an exact sequence

$$0 \to (\mathscr{I} \otimes \Lambda)^W \to (\mathscr{S} \otimes \Lambda)^W \overset{\mapsto}{\to} (\mathscr{S}/\mathscr{I} \otimes \Lambda)^W \to 0.$$

It follows immediately from Solomon's theorem that  $\{dF'_{i_1} \wedge \cdots \wedge dF'_{i_q}: 1 \leq i_1 < \cdots < i_q \leq l\}$  spans  $(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W$  (over **R**). This is in fact a

basis, since  $\mathscr{G}/\mathscr{I}$  affords the regular representation of W, so  $\dim (\mathscr{G}/\mathscr{I} \otimes \Lambda)^W = 2^I$ . We therefore have the following

(3.7) COROLLARY.  $(\mathcal{S}/\mathcal{I}\otimes\Lambda)^W$  is an exterior algebra with generators

$$dF'_i \in [(\mathcal{G}/\mathcal{I})^{m_i} \otimes \Lambda^1]^W$$
, for  $1 \leq i \leq l$ .

We will see later that this exterior algebra is, with degrees in  $\mathscr{G}/\mathscr{I}$  doubled, the cohomology ring of the compact Lie group G. As W-representations, we have  $\mathscr{G}/\mathscr{I} \simeq \mathscr{H}$  and the corollary gives the following

(3.8) MULTIPLICITY FORMULA.

$$\sum_{n=0}^{V} \dim \operatorname{Hom}_{W}(\Lambda^{q}, \mathcal{H}^{n}) u^{n} = s_{q}(u^{m_{1}}, ..., u^{m_{l}}),$$

where  $s_q$  is the elementary symmetric polynomial in l-variables, and the  $m_i$  are the exponents of W.

In particular, the birthday of  $\Lambda^q$  is  $m_1 + \cdots + m_q$ , if g is simple.

(3.9) We close this section with a digression. Suppose g is simple, so all  $\Lambda^q$  are irreducible W-modules. We can actually witness the birth of  $\Lambda^q$  in  $\mathcal{H}$  using the differentials  $dF_i$ , as follows. Choose a basis  $x_1, ..., x_l$  of  $t^*$ , and consider a q-form

$$\omega = \sum f_{i_1, \dots, i_q} dx_{i_1} \wedge \dots \wedge dx_{i_q} \in \mathscr{S} \otimes \Lambda^q.$$

The linear span of the coefficient polynomials  $f_{i_1,...,i_q}$  is independent of the choice of basis  $\{x_i\}$ . Moreover, if  $\omega$  is W-invariant and nonzero, then its coefficients span a W-invariant subspace of  $\mathscr S$  which is isomorphic to  $\Lambda^q$  as a W-module, since the latter is irreducible and self-contragredient.

For example, we have seen that

$$dF_1 \wedge \cdots \wedge dF_l = c \Pi dx_1 \wedge \cdots \wedge dx_q,$$

where c is a nonzero scalar, and  $\Pi$  is the primordial harmonic polynomial, affording the sign character of W. We have a generalization of this for all  $\Lambda^q$ .

(3.10) PROPOSITION. For  $1 \le q \le l$ , the coefficients of  $dF_1 \wedge \cdots \wedge dF_q$  are harmonic polynomials. They span an irreducible W-submodule of  $\mathcal{H}^{m_1 + \cdots + m_q}$ , isomorphic to  $\Lambda^q$ .

*Proof.* The coefficients of  $dF_1 \wedge \cdots \wedge dF_q \in (S^{m_1 + \cdots + m_q} \otimes \Lambda^q)^W$  span a W-invariant subspace of  $S^{m_1 + \cdots + m_q}$ , isomorphic to  $\Lambda^q$ . As in (3.4), these coefficients are harmonic because  $m_1 + \cdots + m_q$  is the birthday of  $\Lambda^q$ , by the multiplicity formula (3.8).  $\square$ 

# 4. Invariant Differential Forms

The ideas in this section go back to E. Cartan and de Rham. For a thorough exposition, see [C-E].

(4.1) Suppose a compact Lie group G acts transitively on a manifold M. Let  $\tau_g$  be the diffeomorphism of M corresponding to  $g \in G$ . A differential p-form  $\omega \in \Omega^p(M)$  is G-invariant if  $\tau_g^* \omega = \omega$ . Such a form is determined by its value at any one point of M. One shows by averaging that every de Rham cohomology class on M is represented by a G-invariant form, and that the subcomplex of invariant forms is preserved by the exterior derivative.

Identify M = G/K where K is the stabilizer of a point  $o \in M$ . We have an orthogonal decomposition  $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$ , where  $\mathfrak{r}$  is the Lie algebra of K. Moreover this decomposition is preserved by Ad(K). For example if G acts on itself by left multiplication then K = 1 and  $\mathfrak{n} = \mathfrak{g}$ . For another example take M = G/T, so K = T and  $\mathfrak{n} = \mathfrak{m}$ . In general,  $\mathfrak{n}$  is naturally identified with the tangent space  $T_o(M)$ , so an invariant form  $\tilde{\omega}$  is determined by the skew-symmetric multilinear map

$$\omega = \tilde{\omega}_o : \mathfrak{n} \times \cdots \times \mathfrak{n} \to \mathbf{R}$$
.

That is,  $\omega \in \Lambda^p \mathfrak{n}^*$ . The invariance of  $\tilde{\omega}$  under K implies the Ad(K)-invariance of  $\omega$ . Conversely, any element  $\omega \in (\Lambda^p \mathfrak{n}^*)^K$  determines a G-invariant form  $\tilde{\omega}$  on M by the formula

$$\widetilde{\omega}_{g+o}((d\tau_g)_o X_1, ..., (d\tau_g)_o X_p) = \omega(X_1, ..., X_p),$$

for  $X_1, ..., X_p \in \mathfrak{n}$  and  $g \in G$ . Thus we may identify the G-invariant p-forms on M with the space  $(\Lambda^p \mathfrak{n}^*)^K$ . In this view, the exterior derivative becomes the map  $\delta : (\Lambda^p \mathfrak{n}^*)^K \to (\Lambda^{p+1} \mathfrak{n}^*)^K$  given by

$$\delta\omega(X_0,...,X_p) = \frac{1}{p+1} \sum_{i < j} (-1)^{i+j} \omega([X_i,X_j]_{\mathfrak{n}},X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_p).$$

Here  $\hat{}$  means the term is omitted, and  $[X_i, X_j]_n$  is the projection of  $[X_i, X_j]$  into n along r. The complex  $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$  computes the de Rham cohomology of M.