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where v is the volume of a fundamental polygon in the upper half plane, and e_i denote the number of elliptic cycles of angles $2\pi/i$. For v, there is a formula due to Humbert. The e_i correspond to conjugacy classes of elements of order i in $PS\Gamma$, these in turn to classes of embeddings of fourth and sixth roots of unity into D; there are formulae for these as well. For an updated presentation of all of this, we refer to [Vi].

Meanwhile, Eichler's somewhat breathtaking «tour de force» has been turned into a standard argument with the calculation of a Tamagawa number as its core. Here is a rough sketch. Denote by G the algebraic group (linear, semisimple, anisotropic) defined over $\mathbf{Z}$ by the norm one elements of $D^\times$; thus, $G(\mathbf{Z}) = S\Gamma$ and $G(\mathbf{R}) = SL_2(\mathbf{R})$. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$ and view $G(\mathbf{Q})$ and $G(\mathbf{Z})$ as subgroups of $G(\mathbf{A})$ via the diagonal embedding. Let

$$C = \prod_{p \text{ prime}} G(\mathbf{Z}_p) \quad \text{and} \quad U = G(\mathbf{R}) \times C \, .$$

Then

$$G(\mathbf{A}) = G(\mathbf{Q})U \quad \text{and} \quad G(\mathbf{Q}) \cap U = G(\mathbf{Z}) \, .$$

This induces a bijection of homogeneous spaces

$$G(\mathbf{A})/G(\mathbf{Q}) \cong U/G(\mathbf{Z}) \, ,$$

preserving the volumes with respect to the Tamagawa measure. Now the volume on the left is, by definition, the Tamagawa number, and equals 1, whence the equation

$$\operatorname{vol}(G(\mathbf{R})/G(\mathbf{Z})) = (\operatorname{vol} C)^{-1} \, .$$

Here, the volume on the right is easy and equals $\zeta(2)\varphi(d)d^{-1}$. The left side can be translated into the volume of a fundamental of $G(\mathbf{Z})$ in the upper half plane, and Gauss-Bonnet brings in the genus. The details can be found in [Vi, ch. IV].

6. Presentations III: K_2

As a byproduct of their computations, Kirchheimer and Wolfart [KW] obtained a description of $K_2(R)$ for the rings R they treated. Conversely, if $K_2(R)$ happens to be known from another source, one can derive presentations of $SL_n(R)$, $n \geqslant 3$. This idea has been pursued in a series of papers by Hurrelbrink ([Hu 1] - [Hu 3]). The general argument runs as follows.

Let R be any commutative ring, $n \geqslant 3$, and for $r \in R$ let $e_{ij}(r)$ be the elementary matrix in $SL_n(R)$ having r in the $i-j$-position $(i \neq j)$. Then we have the "trivial" relations

$$(6) \qquad \begin{cases} e_{ij}(s)e_{ij}(r) & = e_{ij}(s+r), \\ [e_{ij}(s), e_{jl}(r)] & = e_{il}(sr), i \neq l \\ [e_{ij}(s), e_{kl}(r)] & = 1, j \neq k, i \neq l \, . \end{cases}$$

Let $St_n(R)$ be the abstract group generated by elements $x_{ij}(s)$, $s \in R$, with relations as in (6). $St_n(R)$ is called the n-th Steinberg group, and there is an obvious surjective homomorphism

$$\varphi_n = St_n(R) \rightarrow E_n(R) \, ,$$

$E_n(R)$ denoting the subgroup of $SL_n(R)$ generated by the $e_{ij}(r)$. The kernel of φ_n is denoted $K_2(n, R)$. As for GL_2 we can form the direct limit

$$St(R) = \varinjlim St_n(R)$$

and obtain a surjection

$$\varphi = \lim \varphi_n : St(R) \rightarrow E(R) = \lim E_n(R)$$

with kernel

$$K_2(R) = \lim K_2(n, R) \, .$$

Thus $K_2(R)$ codifies the nontrivial relations among elementary matrices over R of *all* sizes. Now let R be the integral domain of a number field. Here we have two stability results: Vaserstein [Va] showed that

$$E_n(R) = SL_n(R), \qquad \text{for } n \geqslant 3 \, ,$$

and van der Kallen [Ka] that

$$K_2(n, r) = K_2(R), \qquad \text{for } n \geqslant 3 \, ,$$

both under the hypothesis that $R^\times$ is infinite, thus excluding $R = \mathbf{Z}$ and the imaginary quadratic case.

Consequently, if one knows generators of $K_2(R)$ in terms of the $x_{ij}(s)$, one can write down immediately presentations of $SL_n(R)$, $n \geqslant 3$. Now how can one possibly know something about $K_2(R)$ without knowing the matrix relations in advance? The miracle happens in form of the Birch-Tate conjecture: assume that $K = \operatorname{Quot} R$ is totally real. Let $\zeta_K(s)$ be the Dedekind zeta function of K. The Birch-Tate conjecture predicts that

$$(7) \qquad \# K_2(R) = w_2(K) \, | \, \zeta_K(-1) \, | \, ,$$

where $w_2(K)$ is a natural number which is easily computed. It follows from the results of [MW] that the odd part of (7) is true if K is abelian. This makes it possible to calculate the odd part of $\# K_2(R)$ in concrete cases: by the Kronecker-Weber theorem, K is a subfield of a cyclotomic field. From this one derives that $\zeta_K(s)$ is a product of Dirichlet series the values of which at negative integers can be expressed by generalized Bernoulli numbers. Finally, the 2-part of $\# K_2(R)$ has been calculated in some real quadratic cases by Browkin and Schinzel [BrS]. Collecting these informations, one has, e.g.,

$$\# K_2(R) = 12 \text{ for } K = \mathbf{Q}(\sqrt{6}) ,$$

([Hu 3], Th. 8). Now it is not too difficult to write down sufficiently many different elements of $K_2(R)$ (so-called Steinberg and Dennis-Stein symbols). Thus, one knows $K_2(R)$, and presentations of $SL_n(R)$, $n \geqslant 3$, drop out. In [Hu 2], Hurrelbrink treats the integral domains of the real subfields of the 9-th and 15-th cyclotomic field, this time relying on the Birch-Tate conjecture for these fields. A generalization of this line of thought to cases involving skew fields seems to be out of sight at present.

I would like to mention here (although K-theory is not explicitely used) a purely algebraic method due to P.M. Cohn [C] which gives presentations of $SL_2(R)$ for certain subrings R of $\mathbf{C}$; this method applies to the integral domains of the euclidean imaginary quadratic fields $\mathbf{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. The presentations involve *all* matrices

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \ y \text{ a unit} ,$$

hence are, by genesis, not finite. In the cases in question it is however possible to reduce them to finite presentations. This is carried out in [F, p. 73 ff.].

7. Cohomology

We recall some notions from the cohomology theory of groups; ideal references for our purposes are the book [Br] by K. Brown and Serre's article [Se 3].

A group Γ is said to have cohomological dimension n, $cd\,\Gamma = n$, if n is the maximal dimension for which there exists a Γ-module M such that $H^n(\Gamma, M) \neq 0$. If there is no such n, $cd\,\Gamma = \infty$. If $cd\,\Gamma < \infty$, then Γ is torsion free. It is known that $cd\,\Gamma = 1$ if and only if Γ is free. There is a virtual notion: $vcd\,\Gamma = n$ if Γ contains a torsion free subgroup Δ of finite index