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**Autor:** Kleinert, Ernst  
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where  $v$  is the volume of a fundamental polygon in the upper half plane, and  $e_i$  denote the number of elliptic cycles of angles  $2\pi/i$ . For  $v$ , there is a formula due to Humbert. The  $e_i$  correspond to conjugacy classes of elements of order  $i$  in  $PS\Gamma$ , these in turn to classes of embeddings of fourth and sixth roots of unity into  $D$ ; there are formulae for these as well. For an updated presentation of all of this, we refer to [Vi].

Meanwhile, Eichler's somewhat breathtaking «tour de force» has been turned into a standard argument with the calculation of a Tamagawa number as its core. Here is a rough sketch. Denote by  $G$  the algebraic group (linear, semisimple, anisotropic) defined over  $\mathbf{Z}$  by the norm one elements of  $D^\times$ ; thus,  $G(\mathbf{Z}) = S\Gamma$  and  $G(\mathbf{R}) = SL_2(\mathbf{R})$ . Let  $\mathbf{A}$  be the adele ring of  $\mathbf{Q}$  and view  $G(\mathbf{Q})$  and  $G(\mathbf{Z})$  as subgroups of  $G(\mathbf{A})$  via the diagonal embedding. Let

$$C = \prod_{p \text{ prime}} G(\mathbf{Z}_p) \quad \text{and} \quad U = G(\mathbf{R}) \times C .$$

Then

$$G(\mathbf{A}) = G(\mathbf{Q})U \quad \text{and} \quad G(\mathbf{Q}) \cap U = G(\mathbf{Z}) .$$

This induces a bijection of homogeneous spaces

$$G(\mathbf{A})/G(\mathbf{Q}) \cong U/G(\mathbf{Z}) ,$$

preserving the volumes with respect to the Tamagawa measure. Now the volume on the left is, by definition, the Tamagawa number, and equals 1, whence the equation

$$\text{vol}(G(\mathbf{R})/G(\mathbf{Z})) = (\text{vol } C)^{-1} .$$

Here, the volume on the right is easy and equals  $\zeta(2)\varphi(d)d^{-1}$ . The left side can be translated into the volume of a fundamental of  $G(\mathbf{Z})$  in the upper half plane, and Gauss-Bonnet brings in the genus. The details can be found in [Vi, ch. IV].

## 6. PRESENTATIONS III: $K_2$

As a byproduct of their computations, Kirchheimer and Wolfart [KW] obtained a description of  $K_2(R)$  for the rings  $R$  they treated. Conversely, if  $K_2(R)$  happens to be known from another source, one can derive presentations of  $SL_n(R)$ ,  $n \geq 3$ . This idea has been pursued in a series of papers by Hurrelbrink ([Hu 1]-[Hu 3]). The general argument runs as follows.

Let  $R$  be any commutative ring,  $n \geq 3$ , and for  $r \in R$  let  $e_{ij}(r)$  be the elementary matrix in  $SL_n(R)$  having  $r$  in the  $i - j$ -position ( $i \neq j$ ). Then we have the “trivial” relations

$$(6) \quad \begin{cases} e_{ij}(s)e_{ij}(r) &= e_{ij}(s+r), \\ [e_{ij}(s), e_{jl}(r)] &= e_{il}(sr), i \neq l \\ [e_{ij}(s), e_{kl}(r)] &= 1, j \neq k, i \neq l. \end{cases}$$

Let  $St_n(R)$  be the abstract group generated by elements  $x_{ij}(s)$ ,  $s \in R$ , with relations as in (6).  $St_n(R)$  is called the  $n$ -th Steinberg group, and there is an obvious surjective homomorphism

$$\varphi_n = St_n(R) \rightarrow E_n(R),$$

$E_n(R)$  denoting the subgroup of  $SL_n(R)$  generated by the  $e_{ij}(r)$ . The kernel of  $\varphi_n$  is denoted  $K_2(n, R)$ . As for  $GL_2$  we can form the direct limit

$$St(R) = \lim_{\rightarrow} St_n(R)$$

and obtain a surjection

$$\varphi = \lim \varphi_n : St(R) \rightarrow E(R) = \lim E_n(R)$$

with kernel

$$K_2(R) = \lim K_2(n, R).$$

Thus  $K_2(R)$  codifies the nontrivial relations among elementary matrices over  $R$  of *all* sizes. Now let  $R$  be the integral domain of a number field. Here we have two stability results: Vaserstein [Va] showed that

$$E_n(R) = SL_n(R), \quad \text{for } n \geq 3,$$

and van der Kallen [Ka] that

$$K_2(n, r) = K_2(R), \quad \text{for } n \geq 3,$$

both under the hypothesis that  $R^\times$  is infinite, thus excluding  $R = \mathbf{Z}$  and the imaginary quadratic case.

Consequently, if one knows generators of  $K_2(R)$  in terms of the  $x_{ij}(s)$ , one can write down immediately presentations of  $SL_n(R)$ ,  $n \geq 3$ . Now how can one possibly know something about  $K_2(R)$  without knowing the matrix relations in advance? The miracle happens in form of the Birch-Tate conjecture: assume that  $K = \text{Quot } R$  is totally real. Let  $\zeta_K(s)$  be the Dedekind zeta function of  $K$ . The Birch-Tate conjecture predicts that

$$(7) \quad \# K_2(R) = w_2(K) |\zeta_K(-1)|,$$

where  $w_2(K)$  is a natural number which is easily computed. It follows from the results of [MW] that the odd part of (7) is true if  $K$  is abelian. This makes it possible to calculate the odd part of  $\# K_2(R)$  in concrete cases: by the Kronecker-Weber theorem,  $K$  is a subfield of a cyclotomic field. From this one derives that  $\zeta_K(s)$  is a product of Dirichlet series the values of which at negative integers can be expressed by generalized Bernoulli numbers. Finally, the 2-part of  $\# K_2(R)$  has been calculated in some real quadratic cases by Browkin and Schinzel [BrS]. Collecting these informations, one has, e.g.,

$$\# K_2(R) = 12 \text{ for } K = \mathbf{Q}(\sqrt{6}),$$

([Hu3], Th. 8). Now it is not too difficult to write down sufficiently many different elements of  $K_2(R)$  (so-called Steinberg and Dennis-Stein symbols). Thus, one knows  $K_2(R)$ , and presentations of  $SL_n(R)$ ,  $n \geq 3$ , drop out. In [Hu2], Hurrelbrink treats the integral domains of the real subfields of the 9-th and 15-th cyclotomic field, this time relying on the Birch-Tate conjecture for these fields. A generalization of this line of thought to cases involving skew fields seems to be out of sight at present.

I would like to mention here (although  $K$ -theory is not explicitly used) a purely algebraic method due to P.M. Cohn [C] which gives presentations of  $SL_2(R)$  for certain subrings  $R$  of  $\mathbf{C}$ ; this method applies to the integral domains of the euclidean imaginary quadratic fields  $\mathbf{Q}(\sqrt{-d})$ ,  $d = 1, 2, 3, 7, 11$ . The presentations involve *all* matrices

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \text{ } y \text{ a unit,}$$

hence are, by genesis, not finite. In the cases in question it is however possible to reduce them to finite presentations. This is carried out in [F, p. 73 ff.].

## 7. COHOMOLOGY

We recall some notions from the cohomology theory of groups; ideal references for our purposes are the book [Br] by K. Brown and Serre's article [Se3].

A group  $\Gamma$  is said to have cohomological dimension  $n$ ,  $cd\Gamma = n$ , if  $n$  is the maximal dimension for which there exists a  $\Gamma$ -module  $M$  such that  $H^n(\Gamma, M) \neq 0$ . If there is no such  $n$ ,  $cd\Gamma = \infty$ . If  $cd\Gamma < \infty$ , then  $\Gamma$  is torsion free. It is known that  $cd\Gamma = 1$  if and only if  $\Gamma$  is free. There is a virtual notion:  $vcd\Gamma = n$  if  $\Gamma$  contains a torsion free subgroup  $\Delta$  of finite index