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Autor: Kleinert, Ernst

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5. Presentations II: indefinite quaternions over the rationals

Suppose that H operates discontinuously on the manifold T. If the operation is in addition fixed-point free, then every $t \in T$ has an open neighbourhood U such that $U \cap Uh = \emptyset$ for $h \neq 1$, and one says that H operates properly discontinuously. The orbit space $X = T/H$ is then a manifold, and if T is simply connected, H is the fundamental group of X. If X belongs to a class of manifolds the fundamental groups of which are known from other sources, then we know H. Using this principle, Eichler [E 1] obtained a description of the unit groups of orders in indefinite quaternion skew fields D over $\mathbf{Q}$. (In the definite case, the unit groups are finite.)

We begin by recalling a few facts from the arithmetic of such D. Let Λ be a maximal order in D. We want to make sure that Γ contains no torsion elements except ± 1. This will be the case if D does not contain the 4-th and 6-th roots of unity (the only ones of degree 2 over $\mathbf{Q}$). For this, it is sufficient that discr Λ contains a prime factor $\equiv 1 \bmod 4$ and one $\equiv 1 \bmod 3$. Namely, let $K = \mathbf{Q}(i)$. Then $K \subset D$ if and only if K splits D. If p is a prime ramified in D (that is, dividing discr Λ), then K splits D at p if and only if $|\mathbf{Q}_p(i) : \mathbf{Q}_p| = 2$, and this is equivalent to $p \equiv 3 \bmod 4$. For the field of 6-th roots of unity, one argues analogously. So we make the above assumption. The only element of order 2 in the norm-one-group $S\Gamma$ is -1 (because if there were another one, it would generate a subfield containing two elements of order 2), and $PS\Gamma = S\Gamma \bmod (\pm 1)$ is torsion free.

By assumption, $D_{\mathbf{R}} \cong M_2(\mathbf{R})$, and the isomorphism maps $S\Gamma$ to a discrete subgroup of $SL_2(\mathbf{R})$. $PS\Gamma$ operates discontinously, and in the well-known manner, on the space $H^+ = SO(2)\backslash SL_2(\mathbf{R})$, which is identified with the upper half-plane. The operation is fixed-point free, because the stabilizer of a point would be in the intersection $SO(2) \cap S\Gamma = (\pm 1)$. Hence $X = H^+/PS\Gamma$ is an oriented surface. By Theorem 1, X is compact. The compact oriented surfaces and their fundamental groups are well-known; we have a presentation

$$PS\Gamma = \pi_1(X) = \langle a_1, b_1, \ldots, a_g, b_g \,|\, \Pi[a_i, b_i] = 1 \rangle .$$

It remains to determine the genus g, which, as the cognoscenti will guess, turns out to be a function of the discriminant. This is accomplished by Eichler (following Hey) with a truly marvellous argument, which we now describe.

Let F_0 be a fundamental domain of $S\Gamma$ in $SL_2(\mathbf{R})$. The cone $C(F_0)$ is then a fundamental domain of $S\Gamma$ in $M_2(\mathbf{R}) = D_{\mathbf{R}}$. Let

$$F = \{x \in C(F_0) \mid -1 \leqslant nr(x) \leqslant 0\} .$$

The idea is to calculate vol F (in Lebesgue measure) in two ways. The first way is to show that vol F is the residue at $s = 1$ of the zeta function of D. This rest (a) on the fact that Λ is a principal ideal domain (see e.g. [R], 35.6), and (b) on a theorem of Dirichlet, which expresses the residue at $S = 1$ of certain functions of "zeta type", associated to a lattice in Euclidean space, by the determinant of the lattice; see [BS], p. 344. Since the zeta function is known (see e.g. [De], p. 130), one gets

$$\operatorname{vol} F = \frac{\pi^2}{12} \frac{\varphi(d)}{d} .$$

(A general formula has been obtained by Käte Hey; cf. the discussion in [De], p. 133.) Here d denotes the fundamental number of D, i.e. the product of the ramified primes, which equals the square root of $|\operatorname{discr} \Lambda|$.

For the second calculation, view D as a cyclic crossed product

$$D = (L \mid \mathbf{Q}, \text{ complex conjugation}) ,$$

$L/\mathbf{Q}$ imaginary quadratic. Then one can write

$$D_{\mathbf{R}} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \mathbf{C} \right\} ,$$

and in this representation $S\Gamma$ operates on the unit circle in $\mathbf{C}$. In the integral for vol F, two of the integrations can be carried out, and there remains an integral over a fundamental domain for $S\Gamma$ in the unit circle, with respect to the invariant measure. But for this, one has the Gauss-Bonnet formula. The final result is

$$g = \frac{\varphi(d)}{12} + 1 .$$

If $S\Gamma$ contain nontrivial torsion elements, one may apply a variant of this reasoning to a torsion-free congruence subgroup.

Soon afterwards, Hull [Hul] gave another treatment, avoiding the analytic argument but making fuller use of the theory of Fuchsian groups; this has the advantage that torsion elements cause no additional problems. The core of the arguments is the genus formula

$$2 - 2g = \upsilon + \frac{1}{2} e_2 + \frac{2}{3} e_3 ,$$

where v is the volume of a fundamental polygon in the upper half plane, and e_i denote the number of elliptic cycles of angles $2\pi/i$. For v, there is a formula due to Humbert. The e_i correspond to conjugacy classes of elements of order i in $PS\Gamma$, these in turn to classes of embeddings of fourth and sixth roots of unity into D; there are formulae for these as well. For an updated presentation of all of this, we refer to [Vi].

Meanwhile, Eichler's somewhat breathtaking «tour de force» has been turned into a standard argument with the calculation of a Tamagawa number as its core. Here is a rough sketch. Denote by G the algebraic group (linear, semisimple, anisotropic) defined over $\mathbf{Z}$ by the norm one elements of $D^\times$; thus, $G(\mathbf{Z}) = S\Gamma$ and $G(\mathbf{R}) = SL_2(\mathbf{R})$. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$ and view $G(\mathbf{Q})$ and $G(\mathbf{Z})$ as subgroups of $G(\mathbf{A})$ via the diagonal embedding. Let

$$C = \prod_{p \text{ prime}} G(\mathbf{Z}_p) \quad \text{and} \quad U = G(\mathbf{R}) \times C .$$

Then

$$G(\mathbf{A}) = G(\mathbf{Q})U \quad \text{and} \quad G(\mathbf{Q}) \cap U = G(\mathbf{Z}) .$$

This induces a bijection of homogeneous spaces

$$G(\mathbf{A})/G(\mathbf{Q}) \cong U/G(\mathbf{Z}) ,$$

preserving the volumes with respect to the Tamagawa measure. Now the volume on the left is, by definition, the Tamagawa number, and equals 1, whence the equation

$$\operatorname{vol}(G(\mathbf{R})/G(\mathbf{Z})) = (\operatorname{vol} C)^{-1} .$$

Here, the volume on the right is easy and equals $\zeta(2)\varphi(d)d^{-1}$. The left side can be translated into the volume of a fundamental of $G(\mathbf{Z})$ in the upper half plane, and Gauss-Bonnet brings in the genus. The details can be found in [Vi, ch. IV].

6. Presentations III: K_2

As a byproduct of their computations, Kirchheimer and Wolfart [KW] obtained a description of $K_2(R)$ for the rings R they treated. Conversely, if $K_2(R)$ happens to be known from another source, one can derive presentations of $SL_n(R)$, $n \geqslant 3$. This idea has been pursued in a series of papers by Hurrelbrink ([Hu 1] - [Hu 3]). The general argument runs as follows.