

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 40 (1994)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: UNITS OF CLASSICAL ORDERS: A SURVEY
Autor: Kleinert, Ernst
Kapitel: 5. PRESENTATIONS II: INDEFINITE QUATERNIONS OVER THE RATIONALS
DOI: <https://doi.org/10.5169/seals-61112>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

5. PRESENTATIONS II: INDEFINITE QUATERNIONS OVER THE RATIONALS

Suppose that H operates discontinuously on the manifold T . If the operation is in addition fixed-point free, then every $t \in T$ has an open neighbourhood U such that $U \cap Uh = \emptyset$ for $h \neq 1$, and one says that H operates properly discontinuously. The orbit space $X = T/H$ is then a manifold, and if T is simply connected, H is the fundamental group of X . If X belongs to a class of manifolds the fundamental groups of which are known from other sources, then we know H . Using this principle, Eichler [E1] obtained a description of the unit groups of orders in indefinite quaternion skew fields D over \mathbf{Q} . (In the definite case, the unit groups are finite.)

We begin by recalling a few facts from the arithmetic of such D . Let Λ be a maximal order in D . We want to make sure that Γ contains no torsion elements except ± 1 . This will be the case if D does not contain the 4-th and 6-th roots of unity (the only ones of degree 2 over \mathbf{Q}). For this, it is sufficient that $\text{discr } \Lambda$ contains a prime factor $\equiv 1 \pmod{4}$ and one $\equiv 1 \pmod{3}$. Namely, let $K = \mathbf{Q}(i)$. Then $K \subset D$ if and only if K splits D . If p is a prime ramified in D (that is, dividing $\text{discr } \Lambda$), then K splits D at p if and only if $|\mathbf{Q}_p(i) : \mathbf{Q}_p| = 2$, and this is equivalent to $p \equiv 3 \pmod{4}$. For the field of 6-th roots of unity, one argues analogously. So we make the above assumption. The only element of order 2 in the norm-one-group $S\Gamma$ is -1 (because if there were another one, it would generate a subfield containing two elements of order 2), and $PS\Gamma = S\Gamma \bmod (\pm 1)$ is torsion free.

By assumption, $D_{\mathbf{R}} \cong M_2(\mathbf{R})$, and the isomorphism maps $S\Gamma$ to a discrete subgroup of $SL_2(\mathbf{R})$. $PS\Gamma$ operates discontinuously, and in the well-known manner, on the space $H^+ = SO(2) \backslash SL_2(\mathbf{R})$, which is identified with the upper half-plane. The operation is fixed-point free, because the stabilizer of a point would be in the intersection $SO(2) \cap S\Gamma = (\pm 1)$. Hence $X = H^+ / PS\Gamma$ is an oriented surface. By Theorem 1, X is compact. The compact oriented surfaces and their fundamental groups are well-known; we have a presentation

$$PS\Gamma = \pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle.$$

It remains to determine the genus g , which, as the cognoscenti will guess, turns out to be a function of the discriminant. This is accomplished by Eichler (following Hey) with a truly marvellous argument, which we now describe.

Let F_0 be a fundamental domain of $S\Gamma$ in $SL_2(\mathbf{R})$. The cone $C(F_0)$ is then a fundamental domain of $S\Gamma$ in $M_2(\mathbf{R}) = D_{\mathbf{R}}$. Let

$$F = \{x \in C(F_0) \mid -1 \leq nr(x) \leq 0\}.$$

The idea is to calculate $\text{vol } F$ (in Lebesgue measure) in two ways. The first way is to show that $\text{vol } F$ is the residue at $s = 1$ of the zeta function of D . This rests (a) on the fact that Λ is a principal ideal domain (see e.g. [R], 35.6), and (b) on a theorem of Dirichlet, which expresses the residue at $S = 1$ of certain functions of "zeta type", associated to a lattice in Euclidean space, by the determinant of the lattice; see [BS], p. 344. Since the zeta function is known (see e.g. [De], p. 130), one gets

$$\text{vol } F = \frac{\pi^2}{12} \frac{\varphi(d)}{d}.$$

(A general formula has been obtained by Käte Hey; cf. the discussion in [De], p. 133.) Here d denotes the fundamental number of D , i.e. the product of the ramified primes, which equals the square root of $|\text{discr } \Lambda|$.

For the second calculation, view D as a cyclic crossed product

$$D = (L \mid \mathbf{Q}, \text{ complex conjugation}),$$

L/\mathbf{Q} imaginary quadratic. Then one can write

$$D_{\mathbf{R}} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} \mid a, b \in \mathbf{C} \right\},$$

and in this representation $S\Gamma$ operates on the unit circle in \mathbf{C} . In the integral for $\text{vol } F$, two of the integrations can be carried out, and there remains an integral over a fundamental domain for $S\Gamma$ in the unit circle, with respect to the invariant measure. But for this, one has the Gauss-Bonnet formula. The final result is

$$g = \frac{\varphi(d)}{12} + 1.$$

If $S\Gamma$ contain nontrivial torsion elements, one may apply a variant of this reasoning to a torsion-free congruence subgroup.

Soon afterwards, Hull [Hul] gave another treatment, avoiding the analytic argument but making fuller use of the theory of Fuchsian groups; this has the advantage that torsion elements cause no additional problems. The core of the arguments is the genus formula

$$2 - 2g = v + \frac{1}{2}e_2 + \frac{2}{3}e_3,$$

where v is the volume of a fundamental polygon in the upper half plane, and e_i denote the number of elliptic cycles of angles $2\pi/i$. For v , there is a formula due to Humbert. The e_i correspond to conjugacy classes of elements of order i in $PS\Gamma$, these in turn to classes of embeddings of fourth and sixth roots of unity into D ; there are formulae for these as well. For an updated presentation of all of this, we refer to [Vi].

Meanwhile, Eichler's somewhat breathtaking «tour de force» has been turned into a standard argument with the calculation of a Tamagawa number as its core. Here is a rough sketch. Denote by G the algebraic group (linear, semisimple, anisotropic) defined over \mathbf{Z} by the norm one elements of D^\times ; thus, $G(\mathbf{Z}) = S\Gamma$ and $G(\mathbf{R}) = SL_2(\mathbf{R})$. Let \mathbf{A} be the adele ring of \mathbf{Q} and view $G(\mathbf{Q})$ and $G(\mathbf{Z})$ as subgroups of $G(\mathbf{A})$ via the diagonal embedding. Let

$$C = \prod_{p \text{ prime}} G(\mathbf{Z}_p) \quad \text{and} \quad U = G(\mathbf{R}) \times C.$$

Then

$$G(\mathbf{A}) = G(\mathbf{Q})U \quad \text{and} \quad G(\mathbf{Q}) \cap U = G(\mathbf{Z}).$$

This induces a bijection of homogeneous spaces

$$G(\mathbf{A})/G(\mathbf{Q}) \cong U/G(\mathbf{Z}),$$

preserving the volumes with respect to the Tamagawa measure. Now the volume on the left is, by definition, the Tamagawa number, and equals 1, whence the equation

$$\text{vol}(G(\mathbf{R})/G(\mathbf{Z})) = (\text{vol } C)^{-1}.$$

Here, the volume on the right is easy and equals $\zeta(2)\varphi(d)d^{-1}$. The left side can be translated into the volume of a fundamental of $G(\mathbf{Z})$ in the upper half plane, and Gauss-Bonnet brings in the genus. The details can be found in [Vi, ch. IV].

6. PRESENTATIONS III: K_2

As a byproduct of their computations, Kirchheimer and Wolfart [KW] obtained a description of $K_2(R)$ for the rings R they treated. Conversely, if $K_2(R)$ happens to be known from another source, one can derive presentations of $SL_n(R)$, $n \geq 3$. This idea has been pursued in a series of papers by Hurrelbrink ([Hu1]-[Hu3]). The general argument runs as follows.