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**Autor:** Kleinert, Ernst  
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## 5. PRESENTATIONS II: INDEFINITE QUATERNIONS OVER THE RATIONALS

Suppose that  $H$  operates discontinuously on the manifold  $T$ . If the operation is in addition fixed-point free, then every  $t \in T$  has an open neighbourhood  $U$  such that  $U \cap Uh = \emptyset$  for  $h \neq 1$ , and one says that  $H$  operates properly discontinuously. The orbit space  $X = T/H$  is then a manifold, and if  $T$  is simply connected,  $H$  is the fundamental group of  $X$ . If  $X$  belongs to a class of manifolds the fundamental groups of which are known from other sources, then we know  $H$ . Using this principle, Eichler [E1] obtained a description of the unit groups of orders in indefinite quaternion skew fields  $D$  over  $\mathbf{Q}$ . (In the definite case, the unit groups are finite.)

We begin by recalling a few facts from the arithmetic of such  $D$ . Let  $\Lambda$  be a maximal order in  $D$ . We want to make sure that  $\Gamma$  contains no torsion elements except  $\pm 1$ . This will be the case if  $D$  does not contain the 4-th and 6-th roots of unity (the only ones of degree 2 over  $\mathbf{Q}$ ). For this, it is sufficient that  $\text{discr } \Lambda$  contains a prime factor  $\equiv 1 \pmod{4}$  and one  $\equiv 1 \pmod{3}$ . Namely, let  $K = \mathbf{Q}(i)$ . Then  $K \subset D$  if and only if  $K$  splits  $D$ . If  $p$  is a prime ramified in  $D$  (that is, dividing  $\text{discr } \Lambda$ ), then  $K$  splits  $D$  at  $p$  if and only if  $|\mathbf{Q}_p(i) : \mathbf{Q}_p| = 2$ , and this is equivalent to  $p \equiv 3 \pmod{4}$ . For the field of 6-th roots of unity, one argues analogously. So we make the above assumption. The only element of order 2 in the norm-one-group  $S\Gamma$  is  $-1$  (because if there were another one, it would generate a subfield containing two elements of order 2), and  $PS\Gamma = S\Gamma \bmod (\pm 1)$  is torsion free.

By assumption,  $D_{\mathbf{R}} \cong M_2(\mathbf{R})$ , and the isomorphism maps  $S\Gamma$  to a discrete subgroup of  $SL_2(\mathbf{R})$ .  $PS\Gamma$  operates discontinuously, and in the well-known manner, on the space  $H^+ = SO(2) \backslash SL_2(\mathbf{R})$ , which is identified with the upper half-plane. The operation is fixed-point free, because the stabilizer of a point would be in the intersection  $SO(2) \cap S\Gamma = (\pm 1)$ . Hence  $X = H^+ / PS\Gamma$  is an oriented surface. By Theorem 1,  $X$  is compact. The compact oriented surfaces and their fundamental groups are well-known; we have a presentation

$$PS\Gamma = \pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle.$$

It remains to determine the genus  $g$ , which, as the cognoscenti will guess, turns out to be a function of the discriminant. This is accomplished by Eichler (following Hey) with a truly marvellous argument, which we now describe.

Let  $F_0$  be a fundamental domain of  $S\Gamma$  in  $SL_2(\mathbf{R})$ . The cone  $C(F_0)$  is then a fundamental domain of  $S\Gamma$  in  $M_2(\mathbf{R}) = D_{\mathbf{R}}$ . Let

$$F = \{x \in C(F_0) \mid -1 \leq nr(x) \leq 0\}.$$

The idea is to calculate  $\text{vol } F$  (in Lebesgue measure) in two ways. The first way is to show that  $\text{vol } F$  is the residue at  $s = 1$  of the zeta function of  $D$ . This rests (a) on the fact that  $\Lambda$  is a principal ideal domain (see e.g. [R], 35.6), and (b) on a theorem of Dirichlet, which expresses the residue at  $s = 1$  of certain functions of "zeta type", associated to a lattice in Euclidean space, by the determinant of the lattice; see [BS], p. 344. Since the zeta function is known (see e.g. [De], p. 130), one gets

$$\text{vol } F = \frac{\pi^2}{12} \frac{\varphi(d)}{d}.$$

(A general formula has been obtained by Käte Hey; cf. the discussion in [De], p. 133.) Here  $d$  denotes the fundamental number of  $D$ , i.e. the product of the ramified primes, which equals the square root of  $|\text{discr } \Lambda|$ .

For the second calculation, view  $D$  as a cyclic crossed product

$$D = (L \mid \mathbf{Q}, \text{ complex conjugation}),$$

$L/\mathbf{Q}$  imaginary quadratic. Then one can write

$$D_{\mathbf{R}} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbf{C} \right\},$$

and in this representation  $S\Gamma$  operates on the unit circle in  $\mathbf{C}$ . In the integral for  $\text{vol } F$ , two of the integrations can be carried out, and there remains an integral over a fundamental domain for  $S\Gamma$  in the unit circle, with respect to the invariant measure. But for this, one has the Gauss-Bonnet formula. The final result is

$$g = \frac{\varphi(d)}{12} + 1.$$

If  $S\Gamma$  contain nontrivial torsion elements, one may apply a variant of this reasoning to a torsion-free congruence subgroup.

Soon afterwards, Hull [Hul] gave another treatment, avoiding the analytic argument but making fuller use of the theory of Fuchsian groups; this has the advantage that torsion elements cause no additional problems. The core of the arguments is the genus formula

$$2 - 2g = v + \frac{1}{2}e_2 + \frac{2}{3}e_3,$$

where  $v$  is the volume of a fundamental polygon in the upper half plane, and  $e_i$  denote the number of elliptic cycles of angles  $2\pi/i$ . For  $v$ , there is a formula due to Humbert. The  $e_i$  correspond to conjugacy classes of elements of order  $i$  in  $PS\Gamma$ , these in turn to classes of embeddings of fourth and sixth roots of unity into  $D$ ; there are formulae for these as well. For an updated presentation of all of this, we refer to [Vi].

Meanwhile, Eichler's somewhat breathtaking «tour de force» has been turned into a standard argument with the calculation of a Tamagawa number as its core. Here is a rough sketch. Denote by  $G$  the algebraic group (linear, semisimple, anisotropic) defined over  $\mathbf{Z}$  by the norm one elements of  $D^\times$ ; thus,  $G(\mathbf{Z}) = S\Gamma$  and  $G(\mathbf{R}) = SL_2(\mathbf{R})$ . Let  $\mathbf{A}$  be the adele ring of  $\mathbf{Q}$  and view  $G(\mathbf{Q})$  and  $G(\mathbf{Z})$  as subgroups of  $G(\mathbf{A})$  via the diagonal embedding. Let

$$C = \prod_{p \text{ prime}} G(\mathbf{Z}_p) \quad \text{and} \quad U = G(\mathbf{R}) \times C.$$

Then

$$G(\mathbf{A}) = G(\mathbf{Q})U \quad \text{and} \quad G(\mathbf{Q}) \cap U = G(\mathbf{Z}).$$

This induces a bijection of homogeneous spaces

$$G(\mathbf{A})/G(\mathbf{Q}) \cong U/G(\mathbf{Z}),$$

preserving the volumes with respect to the Tamagawa measure. Now the volume on the left is, by definition, the Tamagawa number, and equals 1, whence the equation

$$\text{vol}(G(\mathbf{R})/G(\mathbf{Z})) = (\text{vol } C)^{-1}.$$

Here, the volume on the right is easy and equals  $\zeta(2)\varphi(d)d^{-1}$ . The left side can be translated into the volume of a fundamental of  $G(\mathbf{Z})$  in the upper half plane, and Gauss-Bonnet brings in the genus. The details can be found in [Vi, ch. IV].

## 6. PRESENTATIONS III: $K_2$

As a byproduct of their computations, Kirchheimer and Wolfart [KW] obtained a description of  $K_2(R)$  for the rings  $R$  they treated. Conversely, if  $K_2(R)$  happens to be known from another source, one can derive presentations of  $SL_n(R)$ ,  $n \geq 3$ . This idea has been pursued in a series of papers by Hurrelbrink ([Hu1]-[Hu3]). The general argument runs as follows.