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Autor: Kleinert, Ernst

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UNITS OF CLASSICAL ORDERS: A SURVEY

by Ernst KLEINERT

ABSTRACT: This survey describes the principal methods and results in the theory of units of orders.

CONTENTS

1. INTRODUCTION

We consider units of orders in semisimple algebras A of finite dimension over $\mathbf{Q}$. The "algebraic background" for this (a formulation due to Zassenhaus) is the classical theory of algebras, where we find as basic results the Wedderburn decomposition plus the exact sequence

$$1 \to B(K) \to \prod_{\mathfrak{p}} B(K_{\mathfrak{p}}) \to \mathbf{Q}/\mathbf{Z} \to 0 \,,$$

K denoting a number field and $B(K)$ the Brauer group. (For notation and commentary, see [R], § 32). This sequence, which has been called the "Main Theorem in the theory of algebras" ([N], p. 244), in fact contains a full classification of the algebras under consideration and is surely one of the most substantial abbreviations of pure mathematics, incorporating the Hasse norm theorem (exactness on the left for cyclic algebras) as well as the reciprocity law for the norm residue symbol (exactness in the middle) and, implicitly, Hasse's classification of local skew fields (exactness on the right). By an *order* $\Lambda \subset A$ we mean a subring containing 1, consisting of **Z**-integral elements and such that $\mathbf{Q}\Lambda = A$. Natural examples of orders are the integral domains $\mathcal{O}_K$ of number fields K, full matrix rings over $\mathcal{O}_K$, crossed product orders $(\mathcal{O}_L/\mathcal{O}_K, f)$, where L/K is Galois with group G and $f \in H^2(G, \mathcal{O}_L^\times)$ a factor system with values in $\mathcal{O}_L^\times$, and, as a non-simple example, the integral group ring $\mathbf{Z}G$, G finite. The arithmetic of such Λ has two natural parts, namely the theories of modules and of units. The module theory, known as integral representation theory, has been developed systematically and has grown out powerful techniques; lattices over orders enter in various class groups which in turn figure in canonical sequences, or live in almost split sequences which can be arranged to Auslander-Reiten quivers. For maximal orders, the lattice theory can be reduced to a ray class group in the central field, by theorems of Eichler and Swan, and is thus passed to algebraic number theory. In the general case, however, we are at least able to tell why the subject is hopeless: most orders have wildly infinite representation type, which means that their lattices cannot be classified by presently existing methods.

The unit theory is not in such a state, and we still have to subscribe to Eichler's statement in the introduction to his 1935 paper [E1]: "Allein die Einheitentheorie ist noch in keiner Weise abgerundet." There are still very few general results which substantially add to the basic information that unit groups of orders are finitely presentable. This can, of course, not be ascribed to a lack of interest. The point is that classically the interest in integral matrix groups has been concentrated on reduction theory, quotient manifolds and automorphic forms rather than on "the groups themselves." It seems characteristic that neither Siegel in his definite paper [S1] (where he "finished the job", as Weyl put it) nor Weyl in his paraphrase [W] explicitly mention the finite generation, let alone the finite presentation which is effectively proved there; their interest was in finiteness theorems concerning reduction theory of quadratic forms. These classical lines of research have, as everyone knows, been pursued further and have led to the vast and deep generalizations now established in the theory of arithmetic groups, an immense body of methods and results and a meeting ground for more or less the whole apparatus of number theory, algebraic geometry, topology and analysis. Yet it looks strange that for many years no paper seems to have appeared combining the words "units" and "orders" in the title. And it is noteworthy in this connection that most of the results (significant exception: the Bass unit theorem) generalize to larger classes of arithmetic groups and thereby ignore the fact that $\Lambda^\times$ is the unit group of a ring — surely a strong condition on a group. For instance, it should be fruitful to study the natural map $\mathbf{Z}\Lambda^\times \to \Lambda$ from the integral group ring.

The purpose of this survey is to collect the principal methods and theorems about units of orders as far as they refer "more directly" to the structure of these groups (I am aware of the fact that this phrase is not well defined). Scattered and incomplete as the results may be they surely deserve to be presented in some sort of connection. Let us view Dirichlet's unit theorem as a starting point; we will describe three generalizations of it (Theorems 1, 4, 9) which arise from its topological, cohomological, and K-theoretical aspects. In the last section, we present some thoughts about what should be expected from a "General Unit Theorem" which would have satisfied Eichler — certainly a long range project. The reader will also come across a number of more concrete problems which can be attacked with reasonable hope of success.

Any reader will miss something in a survey on a theme which stretches over more or less the whole area of pure mathematics. (I will be grateful to receive criticism as well as hints to further results which fit the theme). On the other side, there will be few to whom I can offer absolutely no news. I readily admit that there are more competent mathematicians who could have written a survey on this subject; however, *non possunt omnes omnia.*

The following notation will be used throughout (unless otherwise specified): A is a semisimple algebra of finite dimension over $\mathbf{Q}$. If A is simple, we write $A = M_n(D)$, D the skewfield part, $K = Z(A) = Z(D)$ the center and $R = \mathcal{O}_K$ the ring of integers of K. $\Lambda \subset A$ is a $\mathbf{Z}$-order (equivalently, R-order) in A, $\Gamma = \Lambda^\times$ the unit group. We exclude from our considerations S-arithmetical and local cases (the former causing complications, the latter being wholly different). Also, we do not treat the specific problems and results for integral group rings which come from the existence of a group base and require special techniques — the isomorphism problem and the Zassenhaus conjecture in its various forms, which are a world of their own and for which I refer to [Ro], and the results due to Hoechsmann, Ritter, Sehgal, and others concerning generators of subgroups of finite index in $(\mathbf{Z}G)^\times$, for which I refer to Sehgal's book [Seh].

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