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**Artikel:** THE THEOREM OF KERÉKJÁRTÓ ON PERIODIC  
HOMEOMORPHISMS OF THE DISC AND THE SPHERE  
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## 4. PERIODIC HOMEOMORPHISMS OF THE SPHERE

The main result of this section is

**THEOREM 4.1.** *Let  $f: S^2 \rightarrow S^2$  be a periodic homeomorphism. Then there exists  $r \in O(3)$  and a homeomorphism  $h: S^2 \rightarrow S^2$  such that  $f = hrh^{-1}$ .*

*Proof of 4.1.* We will divide the proof of Theorem 4.1 into two cases according to whether or not  $f$  has at least one fixed point.

Suppose first that  $f$  has a fixed point. Using 2.5, we deduce the existence of an invariant simple closed curve  $c$  which divides  $S^2$  into two invariant discs  $D_1$  and  $D_2$ .

If  $f$  is orientation preserving and  $f \neq Id$ , then  $f$  has no fixed point on  $c$  (cf. 3.2). Therefore, by Brouwer's fixed point theorem we know then that  $f$  has at least two fixed points; after a conjugacy, we can suppose that  $f$  fixes the two poles  $N$  and  $S$  of  $S^2$ . Using the results of last section, we are able to find  $n$  arcs joining  $N$  and  $S$  such that their union is an invariant set under  $f$ . As in Section 3, we can then construct a conjugacy between  $f$  and a rotation by angle  $2k\pi/n$  around the South-North axis.

If  $f$  is orientation-reversing, then  $f$  has two fixed points on  $c$ . In each of the invariant disc  $D^1$  and  $D^2$ , the fixed point set of  $f$  consists of a simple arc which joins the two fixed points of  $f$  on  $c$ . The union of these two arcs is a simple closed curve which coincides with the fixed point set of  $f$  on  $S^2$ . It is then easy to construct a conjugacy between  $f$  and the reflection about the equator.

Let now suppose that  $f$  has no fixed point on  $S^2$ . Up to conjugacy, we can assume that the second iterate of  $f$ ,  $f^2$  is a periodic rotation around the North-South axis. In particular the points  $N$  and  $S$  are exchanged by  $f$ . For  $t \in (-1, 1)$ , let  $C_t$  be the circle obtained by cutting the sphere by the plane  $z = t$ ,  $D_t$  the disc bordered by  $C_t$  on  $S^2$  which contains  $N$  and:

$$t_0 = \inf\{t \in (-1, 1) ; D_t \cap f(D_t) = \emptyset\}.$$

We write  $D = D_{t_0}$  and  $C = C_{t_0}$  for convenience. Then  $D$  meets  $f(D)$  on its boundary and only on its boundary (see Figure 4). Let  $P_0 \in C \cap f(C)$  and  $P_1, P_2, \dots, P_{n-1}$ , the orbit of  $P_0$  under  $f$ . The points  $P_0, P_2, \dots, P_n$  and  $P_1, P_3, \dots, P_{n-1}$  are distinct because  $f^2$  is a rotation of period  $n/2$ .

Suppose that there exists  $i \in \{1, 3, \dots, n-1\}$  such that  $P_0$  and  $P_i = f^i(P_0)$  coincide. Then  $P_0, S$  and  $N$  are fixed by  $f^{2i}$  so  $f^{2i} = Id$ . Therefore  $2i = n$ .

Let  $b_0$  be the arc of great circle that joins  $N$  to  $P_0$  in  $D$  and  $b_{n/2}$  its image under  $f^{n/2}$ . Then  $b = b_0 \cup b_{n/2}$  is a simple arc joining  $N$  and  $S$  and not meeting its first  $(n/2) - 1$  iterates under  $f$  away from  $N$  and  $S$ . These arcs divide the sphere into  $n/2$  sectors and we can build a conjugacy between  $f$  and the composition of a rotation of period  $n/2$  around the North-South axis with a reflexion about the equator.

Suppose now that the points  $P_0, P_1, \dots, P_{n-1}$  are distinct. Let  $b_0$  an arc of great circle joining  $N$  and  $P_0$  in  $D$  and  $b'_0$  an arc joining  $S$  to  $P_0$  in  $f(D)$  disjoint from  $f(b_0)$  and from its first  $n - 1$  iterates (which is possible since  $f^2$  is a rotation). The union of these two arcs is again a simple arc joining  $N$  and  $S$  which does not meet its first  $n - 1$  iterates under  $f$  away from  $N$  and  $S$ . The union of this arc and its iterates divides the sphere  $S^2$  into  $n$  disjoint sectors. In that case,  $f$  is topologically equivalent to the composition of a rotation of period  $n$  around the North-South axis with a reflexion about the equator.  $\square$

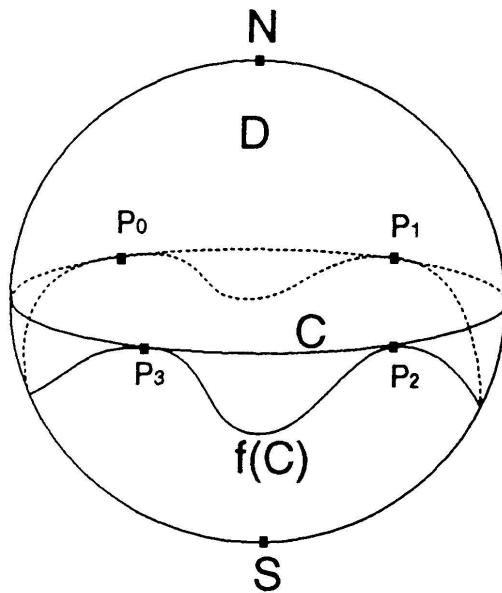


FIGURE 4

**COROLLARY 4.2.** *Let  $f: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be a periodic homeomorphism. Then  $f$  is topologically conjugate to a finite order rotation around the origin or to the reflexion about the  $x$ -axis.*

*Proof of 4.2.* We can extend  $f$  to a homeomorphism of the Sphere  $S^2$  by identifying the plane  $\mathbf{R}^2$  with the complement of the North pole using the stereographic projection. Looking at the proof of 4.1,  $f$  is either equivalent to a rotation around the North-South pole or to a reflexion about a great circle which we can assume to pass through the north pole  $N$ . It is not difficult to

show that the conjugacy can be chosen to fix also the North pole  $N$ . This equivalence induces, therefore, a topological equivalence between  $f$  and a rotation or a reflexion about the  $x$ -axis.  $\square$

*Remark.* The investigation of periodic homeomorphisms on surfaces of positive genus has been studied extensively. We cannot give here a complete bibliography on the subject. We would just like to cite original works of Kerékjártó [4] and Nielsen [13] which lead to the conclusion that a periodic homeomorphism of a Riemannian surface of positive genus is conjugate to a conformal isometry.

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