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4. PERIODIC HOMEOMORPHISMS OF THE SPHERE

The main result of this section is

THEOREM 4.1. *Let $f: S^2 \rightarrow S^2$ be a periodic homeomorphism. Then there exists $r \in O(3)$ and a homeomorphism $h: S^2 \rightarrow S^2$ such that $f = hrh^{-1}$.*

Proof of 4.1. We will divide the proof of Theorem 4.1 into two cases according to whether or not f has at least one fixed point.

Suppose first that f has a fixed point. Using 2.5, we deduce the existence of an invariant simple closed curve c which divides S^2 into two invariant discs D_1 and D_2 .

If f is orientation preserving and $f \neq Id$, then f has no fixed point on c (cf. 3.2). Therefore, by Brouwer's fixed point theorem we know then that f has at least two fixed points; after a conjugacy, we can suppose that f fixes the two poles N and S of S^2 . Using the results of last section, we are able to find n arcs joining N and S such that their union is an invariant set under f . As in Section 3, we can then construct a conjugacy between f and a rotation by angle $2k\pi/n$ around the South-North axis.

If f is orientation-reversing, then f has two fixed points on c . In each of the invariant disc D^1 and D^2 , the fixed point set of f consists of a simple arc which joins the two fixed points of f on c . The union of these two arcs is a simple closed curve which coincides with the fixed point set of f on S^2 . It is then easy to construct a conjugacy between f and the reflection about the equator.

Let now suppose that f has no fixed point on S^2 . Up to conjugacy, we can assume that the second iterate of f , f^2 is a periodic rotation around the North-South axis. In particular the points N and S are exchanged by f . For $t \in (-1, 1)$, let C_t be the circle obtained by cutting the sphere by the plane $z = t$, D_t the disc bordered by C_t on S^2 which contains N and:

$$t_0 = \inf \{ t \in (-1, 1) ; D_t \cap f(D_t) = \emptyset \} .$$

We write $D = D_{t_0}$ and $C = C_{t_0}$ for convenience. Then D meets $f(D)$ on its boundary and only on its boundary (see Figure 4). Let $P_0 \in C \cap f(C)$ and P_1, P_2, \dots, P_{n-1} , the orbit of P_0 under f . The points P_0, P_2, \dots, P_n and P_1, P_3, \dots, P_{n-1} are distinct because f^2 is a rotation of period $n/2$.

Suppose that there exists $i \in \{1, 3, \dots, n-1\}$ such that P_0 and $P_i = f^i(P_0)$ coincide. Then P_0, S and N are fixed by f^{2i} so $f^{2i} = Id$. Therefore $2i = n$.

Let b_0 be the arc of great circle that joins N to P_0 in D and $b_{n/2}$ its image under $f^{n/2}$. Then $b = b_0 \cup b_{n/2}$ is a simple arc joining N and S and not meeting its first $(n/2) - 1$ iterates under f away from N and S . These arcs divide the sphere into $n/2$ sectors and we can build a conjugacy between f and the composition of a rotation of period $n/2$ around the North-South axis with a reflexion about the equator.

Suppose now that the points P_0, P_1, \dots, P_{n-1} are distinct. Let b_0 an arc of great circle joining N and P_0 in D and b'_0 an arc joining S to P_0 in $f(D)$ disjoint from $f(b_0)$ and from its first $n - 1$ iterates (which is possible since f^2 is a rotation). The union of these two arcs is again a simple arc joining N and S which does not meet its first $n - 1$ iterates under f away from N and S . The union of this arc and its iterates divides the sphere S^2 into n disjoint sectors. In that case, f is topologically equivalent to the composition of a rotation of period n around the North-South axis with a reflexion about the equator. \square

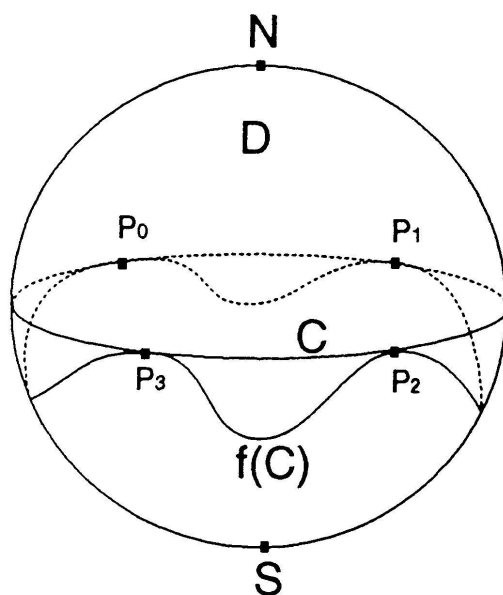


FIGURE 4

COROLLARY 4.2. *Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a periodic homeomorphism. Then f is topologically conjugate to a finite order rotation around the origin or to the reflexion about the x -axis.*

Proof of 4.2. We can extend f to a homeomorphism of the Sphere S^2 by identifying the plane \mathbf{R}^2 with the complement of the North pole using the stereographic projection. Looking at the proof of 4.1, f is either equivalent to a rotation around the North-South pole or to a reflexion about a great circle which we can assume to pass through the north pole N . It is not difficult to

show that the conjugacy can be chosen to fix also the North pole N . This equivalence induces, therefore, a topological equivalence between f and a rotation or a reflexion about the x -axis. \square

Remark. The investigation of periodic homeomorphisms on surfaces of positive genus has been studied extensively. We cannot give here a complete bibliography on the subject. We would just like to cite original works of Kerékjártó [4] and Nielsen [13] which lead to the conclusion that a periodic homeomorphism of a Riemannian surface of positive genus is conjugate to a conformal isometry.

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