

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 40 (1994)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE THEOREM OF KERÉKJÁRTÓ ON PERIODIC
HOMEOMORPHISMS OF THE DISC AND THE SPHERE
Autor: Constantin, Adrian / Kolev, Boris
Kapitel: 3. Periodic Homeomorphisms of the Disc
DOI: <https://doi.org/10.5169/seals-61111>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 04.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The endpoints of γ determine on C_2 an arc δ disjoint from J^o and such that $\delta \cap J = \partial\delta$. We note that there is at most countable family of such arcs γ , noted $(\gamma_i)_{i \in \mathbb{N}}$ and that $\text{diam}(\gamma_i) \rightarrow 0$ as $i \rightarrow \infty$. The boundary of J is the simple closed curve obtained from C_2 when substituting the arcs γ_i for the arcs δ_i and J is a topological disc by the Jordan-Schoenflies theorem. \square

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane \mathbf{R}^2 , namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let $f: S \rightarrow S$ be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold S and let $x \in \text{Fix}(f)$, a fixed point of f . Then for any neighbourhood N of x , there exists a topological disc Δ_x such that:*

1. $\Delta_x \subset N$,
2. Δ_x is a neighbourhood of x ,
3. $f(\Delta_x) = \Delta_x$.

Proof of 2.5. We can first assume that N and its image under $f, f(N)$, are contained in some local chart U homeomorphic with \mathbf{R}^2 and will continue to call x and N the corresponding point and set in \mathbf{R}^2 . Let D_x be an euclidean disc of centre x and radius η where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, \dots, n-1$ and let C_x be its boundary. Let Δ_x be the closure of the component of the invariant set $\bigcap_{k=0}^{n-1} f^k(D_x^o)$ which contains x . By 2.4, Δ_x is a topological disc which is invariant under f (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. \square

Remark. The boundary γ_x of Δ_x , which is an invariant simple closed curve, is contained in $\bigcup_{k=0}^{n-1} f^k(C_x)$.

3. PERIODIC HOMEOMORPHISMS OF THE DISC

THEOREM 3.1. *Let $f: D^2 \rightarrow D^2$ be a periodic homeomorphism. Then there exists $r \in O(2)$ and a homeomorphism $h: D^2 \rightarrow D^2$ such that $f = hrh^{-1}$.*

Before attacking the proof of the result above, let us first look at a special case of Theorem 3.1, namely:

PROPOSITION 3.2. *Let $f: D^2 \rightarrow D^2$ be a periodic homeomorphism such that $f|_{\partial D^2} = Id$. Then $f = Id$.*

Proof of 3.2. Let d be an arbitrary diameter of D^2 with endpoints A and B and let Δ be one of the two connected components of $D^2 - d$. The set:

$$E = \bigcap_{i=1}^n f^i(\Delta^o)$$

is invariant under f and the closure of each of its components is a topological disc.

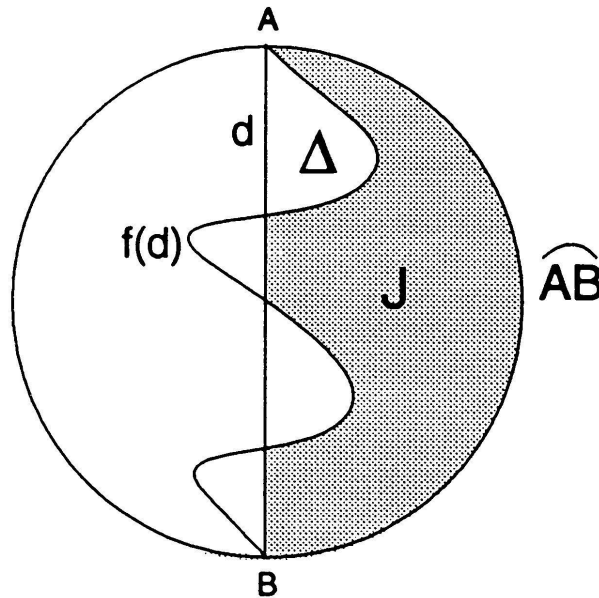


FIGURE 2

Let \widehat{AB} be the arc of circle joining A to B in the boundary of Δ . Since $f^i(\widehat{AB}) = \widehat{AB}$ for all i , there exists a component of E , say J^o , whose closure J contains \widehat{AB} (see Figure 2). By 2.4, J is a topological disc which is invariant under f .

We can write $\partial J = \widehat{AB} \cup \delta$ where δ is an f -invariant, simple arc with endpoints A and B such that:

$$\delta \subset \bigcup_{i=1}^n f^i(d).$$

Since $f(A) = A$ and $f(B) = B$, $f|_{\delta} = Id$. Let x be a point of the arc δ . There exists $i \in \{1, \dots, n\}$ such that $x \in f^i(d)$ and $x = f^{n-i}(x) \in d$ so

that $\delta = d$ and $f|_d = Id$. Since the diameter d was chosen arbitrarily, we have shown that $f = Id$ on D^2 . \square

From now on, f will denote a periodic homeomorphism of the disc of period n with $n > 1$. In the sequel of this section, we prove Theorem 3.1, first investigating the structure of the fixed point set of f .

PROPOSITION 3.3. *Suppose $f: D^2 \mapsto D^2$ is a periodic homeomorphism of period n ($n > 1$); then:*

1. *if f is orientation-preserving, $Fix(f)$ is reduced to a single point which is not on the boundary of D^2 and for $1 \leq i \leq n-1$, $Fix(f^i) = Fix(f)$;*
2. *if f is orientation-reversing, $f^2 = Id$ and $Fix(f)$ is a simple arc which divides D^2 into two topological discs which are permuted by f .*

Proof of 3.3. Suppose first that f is orientation-preserving. By Brouwer fixed point theorem, f has at least one fixed point. Since $f|_{\partial D^2}$ is orientation-preserving and periodic, f has no fixed point on ∂D^2 . Otherwise f would be the identity map on ∂D^2 and using 3.2, f would be the identity map on the whole disc which is excluded by hypothesis. Therefore, f has at least one fixed point in $D^2 \setminus \partial D^2$ which we can assume to be, up to conjugacy, O , the center of the disc.

Let $A = D^2 \setminus \{O\}$. A is a half open annulus which is invariant under f . Suppose now that an iterate f^i of f has a fixed point $x_0 \in A$. Let \tilde{x}_0 be a lift of x_0 to the universal covering space \tilde{A} of A and G be the lift of f^i such that $G(\tilde{x}_0) = \tilde{x}_0$. G^n is a lift of Id which fixes one point, thus $G^n = Id$. In particular, $G|_{\partial \tilde{A}}$ is a periodic and orientation preserving homeomorphism of the line, thus $G = Id$ on $\partial \tilde{A}$. Therefore, $f^i = Id$ on ∂D^2 and, according to 3.2, $f^i = Id$ on the whole disc, so that i is a multiple of n according to the definition of n .

Suppose now that f is orientation-reversing. In that case, f has exactly two fixed points on ∂D^2 which we denote by A and B and f^2 is the identity map on ∂D^2 , therefore, by 3.2, $f^2 = Id$ on D^2 .

We assert that $Fix(f)$ is connected. For if not, we can find two nonempty compact sets K_1 and K_2 such that

$$Fix(f) = K_1 \cup K_2, \quad K_1 \cap K_2 = \emptyset.$$

If $A \in K_1$ and $B \in K_2$, it is then possible to construct a simple arc γ in $D^2 \setminus (K_1 \cup K_2)$ which intersect ∂D^2 only on its endpoints and which

separates A from B . Using the same argument as the one used in the proof of 3.2, we can show the existence of an f -invariant simple arc:

$$\delta \subset \bigcup_{i=0}^{n-1} f^i(\gamma) \subset D^2 \setminus \text{Fix}(f)$$

which separates A from B . But f must then have a fixed point on δ which gives a contradiction. Therefore we can suppose that one of the two compact sets, say K_1 is contained in $D^2 \setminus \partial D^2$. In that case, it is possible to construct a simple closed curve $c \subset D^2 \setminus \partial D^2$ which does not meet $K_1 \cup K_2$ and such that the topological disc it bounds contains at least one point of K_1 . Using similar arguments as those of the proof of 2.5, we can find an f -invariant topological disc in $D^2 \setminus \partial D^2$ whose boundary contains no fixed point. This gives again a contradiction, since any simple closed curve which bounds an invariant disc has exactly two fixed points of f .

The previous arguments applied to an arbitrarily small invariant topological disc around a fixed point given by 2.5 shows that $\text{Fix}(f)$ is also locally connected and by 2.2, $\text{Fix}(f)$ is therefore pathwise connected. In view of 2.1, there exists a simple arc γ in $\text{Fix}(f)$ which joins A and B . This arc divides D^2 into two topological discs Δ_1 and Δ_2 by the Jordan-Schoenflies theorem. $D^2 \setminus \gamma$ is obviously invariant under f and the two arcs on ∂D^2 delimited by A and B are permuted by f , therefore $f(\Delta_1) = \Delta_2$, $f(\Delta_2) = \Delta_1$ and $\text{Fix}(f)$ is reduced to γ . \square

Proof of 3.1. Suppose first that f is orientation-preserving. By 3.3, we can suppose that $\text{Fix}(f) = \{O\}$, the center of the disc. Since $f|_{\partial D^2}$ is a periodic homeomorphism of period n , the rotation number of $f|_{\partial D^2}$, $\rho(f|_{\partial D^2}) = k/n$, where k and n are coprime. We are going to prove that f is conjugate to a rotation by angle $2k\pi/n$ around the origin. Without loss of generality, we can assume that $k = 1$. Indeed, suppose the result holds if $\rho(f|_{\partial D^2}) = 1/n$. Then, if $k > 1$ we replace f by f^j where $j \in \mathbf{N}$ is such that $jk \equiv 1 \pmod{n}$. Then $\rho(f^j|_{\partial D^2}) = 1/n$, thus f^j is conjugate to a rotation by angle $2\pi/n$ around the origin and since $(f^j)^k = f$, it follows that f is conjugate to a rotation by angle $2k\pi/n$.

Let us consider the quotient space D^2/f where two points are identified if they belong to the same orbit under f . D^2/f is endowed with the quotient topology. It is a compact and pathwise connected metric space, the metric being defined by:

$$d(\pi(x), \pi(y)) = \inf_{0 \leq h, k \leq n-1} \{d(f^k(x), f^h(y))\},$$

where $\pi : D^2 \rightarrow D^2/f$ is the canonical projection.

By 2.1, we can find a simple arc γ from $\pi(O)$ to an arbitrary point on $\pi(\partial D^2)$. Since the group of homeomorphisms generated by f acts freely on D^2 except at O it follows that $\pi : D^2 \rightarrow D^2/f$ is a regular branched covering (see [10] page 49). Therefore, $\pi^{-1}(\gamma)$ is the union of n disjoint simple arcs (with the exception of their common endpoint O) $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$, which divide D^2 into n disjoint sectors, A_0, A_1, \dots, A_{n-1} . The hypothesis $\rho(f/\partial D^2) = 1/n$ implies that $\gamma_i = f^i(\gamma_0)$.

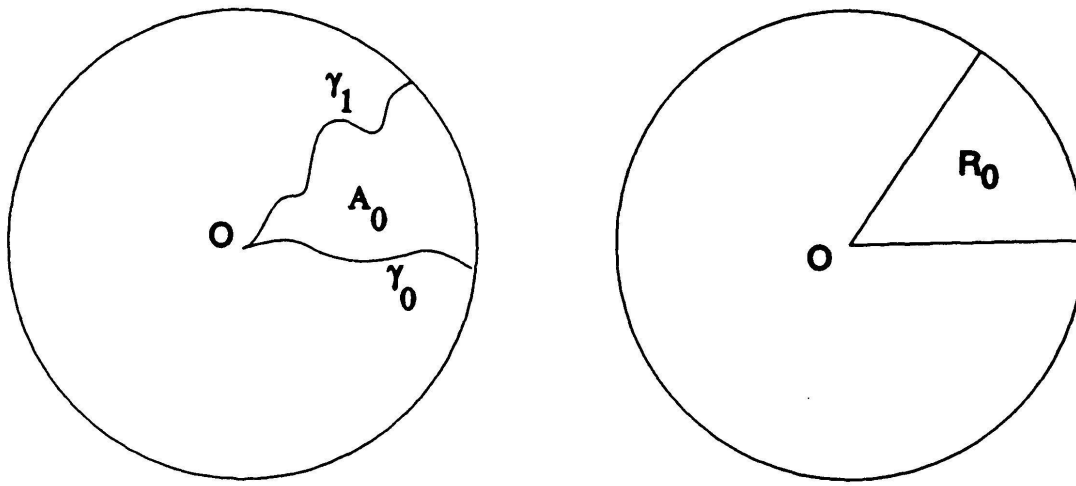


FIGURE 3

Let h be a homeomorphism between A_0 and R_0 , the fundamental region in D^2 of the rotation by angle $2\pi/n$ around the origin, and such that $h \setminus \gamma_1 = rh \setminus \gamma_0$. We can extend h to a homeomorphism of D^2 by defining h/A_i as $r^i h f^{-i}$, r being the rotation of centre O and angle $2\pi/n$. It is easy to verify that h is an homeomorphism of D^2 and that $f = h^{-1}rh$.

Suppose now that f is orientation-reversing. By 3.3, $\text{Fix}(f)$ is a simple arc γ which divides D^2 into two topological discs Δ_1 and Δ_2 which are permuted by f . Let h be a homeomorphism between Δ_1 and the upper half disc D_1 . We define h on Δ_2 in the following way:

$$h(y) = Sh_{/\Delta_1} f(y), \quad y \in \Delta_2,$$

where S is the reflection about the x -axis. It is then easy to verify that h is a homeomorphism of D^2 and this gives a conjugacy between f and S . \square

Remark. Using 3.1, it can also be shown that any periodic homeomorphism of the annulus is topologically equivalent to an euclidean isometry (modulo a flip of the boundary if it is not boundary-preserving).