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**Artikel:** THE THEOREM OF KERÉKJÁRTÓ ON PERIODIC HOMEOMORPHISMS OF THE DISC AND THE SPHERE

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**Kapitel:** 2. Background and Definitions

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## 2. Background and Definitions

Let $X$ be a topological space and $f$ a homeomorphism of $X$. We say that $f$ is periodic if there is an integer $n > 0$ such that $f^n = Id$. The period of $f$ is the smallest positive integer $n$ with this property.

As we will use them without further justifications, let us first recall some basic properties of one-dimensional maps.

Let $f: I \to I$ be a periodic homeomorphism of the unit interval. If $f$ preserves the endpoints then $f$ is the identity map. If $f$ exchanges the endpoints then $f^2 = Id$ and $f$ is conjugate to the reflection map $x \mapsto 1 - x$. Similarly, a periodic homeomorphism of the real line **R** is the identity map or is a conjugate of the involution $x \mapsto -x$ according to whether it is an increasing or a decreasing function.

Let $f: S^1 \to S^1$ be a periodic homeomorphism of period $n$ of the unit circle. If $f$ is order-preserving then the rotation number of $f$, $\rho(f) = k/n$, where $k$ and $n$ are coprime (see [5] for an excellent exposition on rotation numbers) and $f$ is conjugate to a rotation of angle $2k\pi/n$. If $f$ is order-reversing then $f$ has exactly two fixed points, $f^2$ is the identity map and the two arcs delimited on $S^1$ by the fixed points of $f$ are permuted by $f$.

A metric space $X$ is path connected if there exists a continuous map from the unit interval [0, 1] into $X$ which joins any two given points. It is arcwise connected if there is a topological embedding of [0, 1] into $X$ which joins any two given distinct points. In fact, it can be shown that the two notions are equivalent (see [14, Theorem 4.1] or [11, Lemma 16.3]).

LEMMA 2.1. *A metric space $X$ is path connected if and only if it is arcwise connected.*

A useful characterisation of path connected spaces is given in term of local connectivity. A metric space $X$ is locally connected if each point of $X$ possesses arbitrary small connected neighbourhoods. The following can be shown (see [8, Theorem 3.15] or [11, Lemma 16.4]):

LEMMA 2.2. *A compact, connected and locally connected metric space is pathwise connected.*

Another important ingredient used in this article, and in fact the ultimate result we will need, is the famous Jordan-Schoenflies theorem on simple closed curves in the plane (see [2, 9] or [12, Theorem 17.1]).

THEOREM 2.3 (Jordan-Schoenflies). *Every simple closed curve $J$ divides the plane into exactly two components of each of which it is the* *complete boundary and the closure of the bounded component can be mapped topologically onto the closed unit disc.*

In what follows, a *closed* topological disc (or just a topological disc) $D$ is the image under a topological embedding of the *closed* unit disc and we write $D^o$ for its interior and $\partial D$ for its boundary. However, the closure of a bounded open set which is homeomorphic to the open unit disc is not necessarily a closed topological disc [11, Chapter 15].

PROPOSITION 2.4. *Let* $D_1, D_2, \ldots, D_n$ *be a finite number of closed topological discs in the plane and* $J^o$ *be any connected component of* $\bigcap_{i=1}^n D_i^o$. *Then* $\partial J$ *is a simple closed curve and* $J$ *the closure of* $J^o$ *is a topological disc.*

*Proof of 2.4.* We will use induction on $n$, the number of discs. If $n = 1$ this is just the Jordan-Schoenflies theorem, so let us suppose that the result holds for some $n (n \geqslant 1)$ and let $J^o$ be any component of the complement of $n + 1$ topological discs $D_1, D_2, \ldots, D_{n+1}$ in the plane. Let $K^o$ be the component of $\bigcap_{i=1}^n D_i^o$ that contains $J^o$. By induction, its closure $K$ is a topological disc. Since $J^o$ is a component of $K^o \cap D_{n+1}^o$, it suffices to show that the result holds for two discs $D_1$ and $D_2$ (see Figure 1). Set $C_i = \partial D_i$ for $i = 1, 2$ and let $J$ be the closure of a component of $D_1^o \cap D_2^o$. We have that $\partial J \neq \emptyset$ and $\partial J \subset C_1 \cup C_2$. If $\partial J$ is entirely contained in one of the two curves, say $C_1$, then $J = D_1$ and the lemma is proved. We can thus suppose that $\partial J \not\subset C_1$ and $\partial J \not\subset C_2$.

Let $x \in \partial J$, $x \notin C_2$. Then $x \in C_1 \cap D_2^o$, and we can find an arc $\gamma$ in $C_1$ such that:

$$x \in \gamma, \quad \gamma \subset \partial J, \quad \gamma \backslash \partial\gamma \subset D_2^o, \quad \partial\gamma \subset C_2 .$$

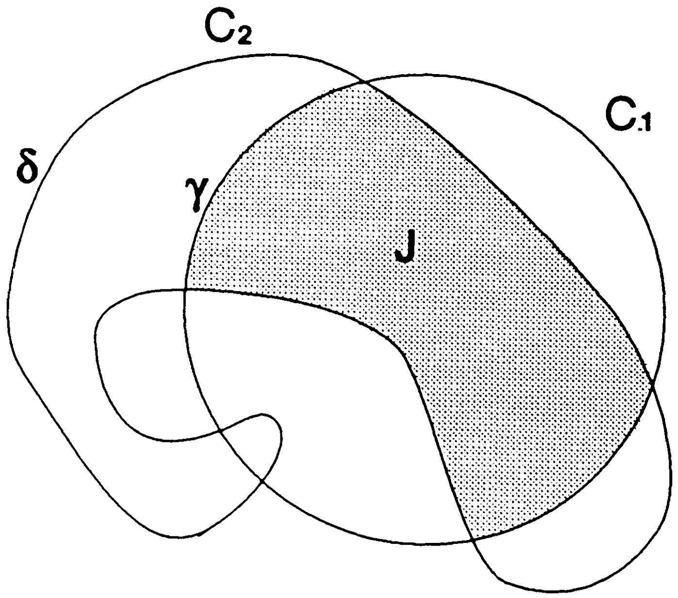


FIGURE 1

The endpoints of $\gamma$ determine on $C_2$ an arc $\delta$ disjoint from $J^o$ and such that $\delta \cap J = \partial\delta$. We note that there is an at most countable family of such arcs $\gamma$, noted $(\gamma_i)_{i \in N}$ and that $diam(\gamma_i) \to 0$ as $i \to \infty$. The boundary of $J$ is the simple closed curve obtained from $C_2$ when substituting the arcs $\gamma_i$ for the arcs $\delta_i$ and $J$ is a topological disc by the Jordan-Schoenflies theorem. □

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane $\mathbf{R}^2$, namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let* $f: S \to S$ *be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold* $S$ *and let* $x \in Fix(f)$, *a fixed point of* $f$. *Then for any neighbourhood* $N$ *of* $x$, *there exists a topological disc* $\Delta_x$ *such that:*

*1.* $\Delta_x \subset N$,

*2.* $\Delta_x$ *is a neighbourhood of* $x$,

*3.* $f(\Delta_x) = \Delta_x$.

*Proof of 2.5.* We can first assume that $N$ and its image under $f$, $f(N)$, are contained in some local chart $U$ homeomorphic with $\mathbf{R}^2$ and will continue to call $x$ and $N$ the corresponding point and set in $\mathbf{R}^2$. Let $D_x$ be an euclidean disc of centre $x$ and radius $\eta$ where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, .., n-1$ and let $C_x$ be its boundary. Let $\Delta_x$ be the closure of the component of the invariant set $\cap_{k=0}^{n-1} f^k(D_x^o)$ which contains $x$. By 2.4, $\Delta_x$ is a topological disc which is invariant under $f$ (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. □

*Remark.* The boundary $\gamma_x$ of $\Delta_x$, which is an invariant simple closed curve, is contained in $\cup_{k=0}^{n-1} f^k(C_x)$.

## 3. Periodic Homeomorphisms of the Disc

THEOREM 3.1. *Let* $f: D^2 \to D^2$ *be a periodic homeomorphism. Then there exists* $r \in O(2)$ *and a homeomorphism* $h: D^2 \to D^2$ *such that* $f = hrh^{-1}$.