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2. BACKGROUND AND DEFINITIONS

Let X be a topological space and f a homeomorphism of X . We say that f is periodic if there is an integer $n > 0$ such that $f^n = Id$. The period of f is the smallest positive integer n with this property.

As we will use them without further justifications, let us first recall some basic properties of one-dimensional maps.

Let $f: I \rightarrow I$ be a periodic homeomorphism of the unit interval. If f preserves the endpoints then f is the identity map. If f exchanges the endpoints then $f^2 = Id$ and f is conjugate to the reflection map $x \mapsto 1 - x$. Similarly, a periodic homeomorphism of the real line \mathbf{R} is the identity map or is a conjugate of the involution $x \mapsto -x$ according to whether it is an increasing or a decreasing function.

Let $f: S^1 \rightarrow S^1$ be a periodic homeomorphism of period n of the unit circle. If f is order-preserving then the rotation number of f , $\rho(f) = k/n$, where k and n are coprime (see [5] for an excellent exposition on rotation numbers) and f is conjugate to a rotation of angle $2k\pi/n$. If f is order-reversing then f has exactly two fixed points, f^2 is the identity map and the two arcs delimited on S^1 by the fixed points of f are permuted by f .

A metric space X is path connected if there exists a continuous map from the unit interval $[0, 1]$ into X which joins any two given points. It is arcwise connected if there is a topological embedding of $[0, 1]$ into X which joins any two given distinct points. In fact, it can be shown that the two notions are equivalent (see [14, Theorem 4.1] or [11, Lemma 16.3]).

LEMMA 2.1. *A metric space X is path connected if and only if it is arcwise connected.*

A useful characterisation of path connected spaces is given in term of local connectivity. A metric space X is locally connected if each point of X possesses arbitrary small connected neighbourhoods. The following can be shown (see [8, Theorem 3.15] or [11, Lemma 16.4]):

LEMMA 2.2. *A compact, connected and locally connected metric space is pathwise connected.*

Another important ingredient used in this article, and in fact the ultimate result we will need, is the famous Jordan-Schoenflies theorem on simple closed curves in the plane (see [2, 9] or [12, Theorem 17.1]).

THEOREM 2.3 (Jordan-Schoenflies). *Every simple closed curve J divides the plane into exactly two components of each of which it is the*

complete boundary and the closure of the bounded component can be mapped topologically onto the closed unit disc.

In what follows, a *closed* topological disc (or just a topological disc) D is the image under a topological embedding of the *closed* unit disc and we write D° for its interior and ∂D for its boundary. However, the closure of a bounded open set which is homeomorphic to the open unit disc is not necessarily a closed topological disc [11, Chapter 15].

PROPOSITION 2.4. *Let D_1, D_2, \dots, D_n be a finite number of closed topological discs in the plane and J° be any connected component of $\bigcap_{i=1}^n D_i^\circ$. Then ∂J is a simple closed curve and J the closure of J° is a topological disc.*

Proof of 2.4. We will use induction on n , the number of discs. If $n = 1$ this is just the Jordan-Schoenflies theorem, so let us suppose that the result holds for some $n (n \geq 1)$ and let J° be any component of the complement of $n + 1$ topological discs D_1, D_2, \dots, D_{n+1} in the plane. Let K° be the component of $\bigcap_{i=1}^n D_i^\circ$ that contains J° . By induction, its closure K is a topological disc. Since J° is a component of $K^\circ \cap D_{n+1}^\circ$, it suffices to show that the result holds for two discs D_1 and D_2 (see Figure 1). Set $C_i = \partial D_i$ for $i = 1, 2$ and let J be the closure of a component of $D_1^\circ \cap D_2^\circ$. We have that $\partial J \neq \emptyset$ and $\partial J \subset C_1 \cup C_2$. If ∂J is entirely contained in one of the two curves, say C_1 , then $J = D_1$ and the lemma is proved. We can thus suppose that $\partial J \not\subset C_1$ and $\partial J \not\subset C_2$.

Let $x \in \partial J$, $x \notin C_2$. Then $x \in C_1 \cap D_2^\circ$, and we can find an arc γ in C_1 such that:

$$x \in \gamma, \quad \gamma \subset \partial J, \quad \gamma \setminus \partial \gamma \subset D_2^\circ, \quad \partial \gamma \subset C_2.$$

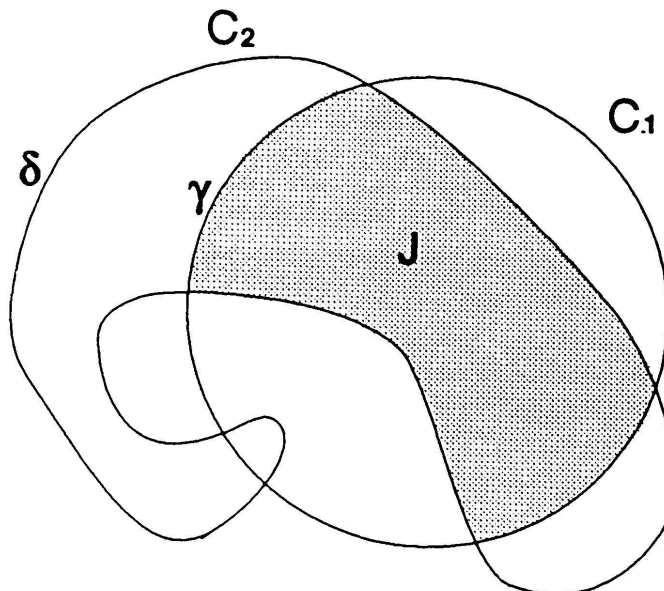


FIGURE 1

The endpoints of γ determine on C_2 an arc δ disjoint from J^o and such that $\delta \cap J = \partial\delta$. We note that there is an at most countable family of such arcs γ , noted $(\gamma_i)_{i \in N}$ and that $\text{diam}(\gamma_i) \rightarrow 0$ as $i \rightarrow \infty$. The boundary of J is the simple closed curve obtained from C_2 when substituting the arcs γ_i for the arcs δ_i and J is a topological disc by the Jordan-Schoenflies theorem. \square

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane \mathbf{R}^2 , namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let $f: S \rightarrow S$ be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold S and let $x \in \text{Fix}(f)$, a fixed point of f . Then for any neighbourhood N of x , there exists a topological disc Δ_x such that:*

1. $\Delta_x \subset N$,
2. Δ_x is a neighbourhood of x ,
3. $f(\Delta_x) = \Delta_x$.

Proof of 2.5. We can first assume that N and its image under $f, f(N)$, are contained in some local chart U homeomorphic with \mathbf{R}^2 and will continue to call x and N the corresponding point and set in \mathbf{R}^2 . Let D_x be an euclidean disc of centre x and radius η where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, \dots, n-1$ and let C_x be its boundary. Let Δ_x be the closure of the component of the invariant set $\bigcap_{k=0}^{n-1} f^k(D_x^o)$ which contains x . By 2.4, Δ_x is a topological disc which is invariant under f (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. \square

Remark. The boundary γ_x of Δ_x , which is an invariant simple closed curve, is contained in $\bigcup_{k=0}^{n-1} f^k(C_x)$.

3. PERIODIC HOMEOMORPHISMS OF THE DISC

THEOREM 3.1. *Let $f: D^2 \rightarrow D^2$ be a periodic homeomorphism. Then there exists $r \in O(2)$ and a homeomorphism $h: D^2 \rightarrow D^2$ such that $f = hrh^{-1}$.*