

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 40 (1994)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE THEOREM OF KERÉKJÁRTÓ ON PERIODIC
HOMEOMORPHISMS OF THE DISC AND THE SPHERE
Autor: Constantin, Adrian / Kolev, Boris
Kapitel: 2. Background and Definitions
DOI: <https://doi.org/10.5169/seals-61111>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 24.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2. BACKGROUND AND DEFINITIONS

Let X be a topological space and f a homeomorphism of X . We say that f is periodic if there is an integer $n > 0$ such that $f^n = Id$. The period of f is the smallest positive integer n with this property.

As we will use them without further justifications, let us first recall some basic properties of one-dimensional maps.

Let $f: I \rightarrow I$ be a periodic homeomorphism of the unit interval. If f preserves the endpoints then f is the identity map. If f exchanges the endpoints then $f^2 = Id$ and f is conjugate to the reflection map $x \mapsto 1 - x$. Similarly, a periodic homeomorphism of the real line \mathbf{R} is the identity map or is a conjugate of the involution $x \mapsto -x$ according to whether it is an increasing or a decreasing function.

Let $f: S^1 \rightarrow S^1$ be a periodic homeomorphism of period n of the unit circle. If f is order-preserving then the rotation number of f , $\rho(f) = k/n$, where k and n are coprime (see [5] for an excellent exposition on rotation numbers) and f is conjugate to a rotation of angle $2k\pi/n$. If f is order-reversing then f has exactly two fixed points, f^2 is the identity map and the two arcs delimited on S^1 by the fixed points of f are permuted by f .

A metric space X is path connected if there exists a continuous map from the unit interval $[0, 1]$ into X which joins any two given points. It is arcwise connected if there is a topological embedding of $[0, 1]$ into X which joins any two given distinct points. In fact, it can be shown that the two notions are equivalent (see [14, Theorem 4.1] or [11, Lemma 16.3]).

LEMMA 2.1. *A metric space X is path connected if and only if it is arcwise connected.*

A useful characterisation of path connected spaces is given in term of local connectivity. A metric space X is locally connected if each point of X possesses arbitrary small connected neighbourhoods. The following can be shown (see [8, Theorem 3.15] or [11, Lemma 16.4]):

LEMMA 2.2. *A compact, connected and locally connected metric space is pathwise connected.*

Another important ingredient used in this article, and in fact the ultimate result we will need, is the famous Jordan-Schoenflies theorem on simple closed curves in the plane (see [2, 9] or [12, Theorem 17.1]).

THEOREM 2.3 (Jordan-Schoenflies). *Every simple closed curve J divides the plane into exactly two components of each of which it is the*

complete boundary and the closure of the bounded component can be mapped topologically onto the closed unit disc.

In what follows, a *closed* topological disc (or just a topological disc) D is the image under a topological embedding of the *closed* unit disc and we write D° for its interior and ∂D for its boundary. However, the closure of a bounded open set which is homeomorphic to the open unit disc is not necessarily a closed topological disc [11, Chapter 15].

PROPOSITION 2.4. *Let D_1, D_2, \dots, D_n be a finite number of closed topological discs in the plane and J° be any connected component of $\cap_{i=1}^n D_i^\circ$. Then ∂J is a simple closed curve and J the closure of J° is a topological disc.*

Proof of 2.4. We will use induction on n , the number of discs. If $n = 1$ this is just the Jordan-Schoenflies theorem, so let us suppose that the result holds for some $n (n \geq 1)$ and let J° be any component of the complement of $n + 1$ topological discs D_1, D_2, \dots, D_{n+1} in the plane. Let K° be the component of $\cap_{i=1}^n D_i^\circ$ that contains J° . By induction, its closure K is a topological disc. Since J° is a component of $K^\circ \cap D_{n+1}^\circ$, it suffices to show that the result holds for two discs D_1 and D_2 (see Figure 1). Set $C_i = \partial D_i$ for $i = 1, 2$ and let J be the closure of a component of $D_1^\circ \cap D_2^\circ$. We have that $\partial J \neq \emptyset$ and $\partial J \subset C_1 \cup C_2$. If ∂J is entirely contained in one of the two curves, say C_1 , then $J = D_1$ and the lemma is proved. We can thus suppose that $\partial J \not\subset C_1$ and $\partial J \not\subset C_2$.

Let $x \in \partial J$, $x \notin C_2$. Then $x \in C_1 \cap D_2^\circ$, and we can find an arc γ in C_1 such that:

$$x \in \gamma, \quad \gamma \subset \partial J, \quad \gamma \setminus \partial \gamma \subset D_2^\circ, \quad \partial \gamma \subset C_2.$$

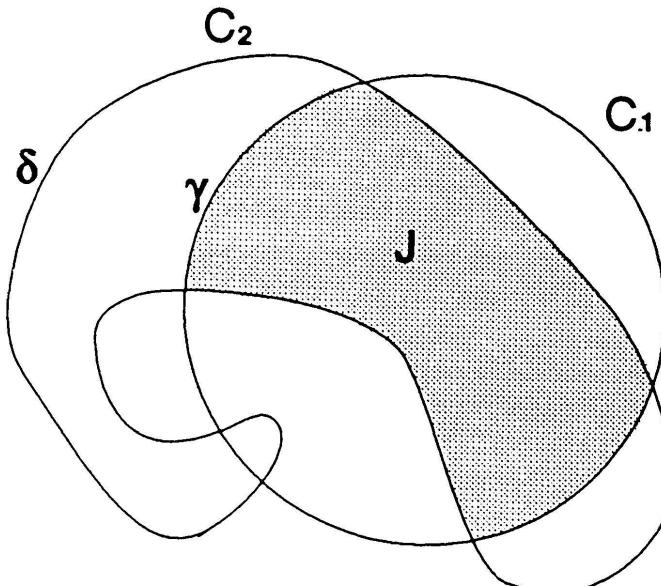


FIGURE 1

The endpoints of γ determine on C_2 an arc δ disjoint from J^o and such that $\delta \cap J = \partial\delta$. We note that there is an at most countable family of such arcs γ , noted $(\gamma_i)_{i \in N}$ and that $\text{diam}(\gamma_i) \rightarrow 0$ as $i \rightarrow \infty$. The boundary of J is the simple closed curve obtained from C_2 when substituting the arcs γ_i for the arcs δ_i and J is a topological disc by the Jordan-Schoenflies theorem. \square

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane \mathbf{R}^2 , namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let $f: S \rightarrow S$ be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold S and let $x \in \text{Fix}(f)$, a fixed point of f . Then for any neighbourhood N of x , there exists a topological disc Δ_x such that:*

1. $\Delta_x \subset N$,
2. Δ_x is a neighbourhood of x ,
3. $f(\Delta_x) = \Delta_x$.

Proof of 2.5. We can first assume that N and its image under f , $f(N)$, are contained in some local chart U homeomorphic with \mathbf{R}^2 and will continue to call x and N the corresponding point and set in \mathbf{R}^2 . Let D_x be an euclidean disc of centre x and radius η where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, \dots, n - 1$ and let C_x be its boundary. Let Δ_x be the closure of the component of the invariant set $\bigcap_{k=0}^{n-1} f^k(D_x^o)$ which contains x . By 2.4, Δ_x is a topological disc which is invariant under f (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. \square

Remark. The boundary γ_x of Δ_x , which is an invariant simple closed curve, is contained in $\bigcup_{k=0}^{n-1} f^k(C_x)$.

3. PERIODIC HOMEOMORPHISMS OF THE DISC

THEOREM 3.1. *Let $f: D^2 \rightarrow D^2$ be a periodic homeomorphism. Then there exists $r \in O(2)$ and a homeomorphism $h: D^2 \rightarrow D^2$ such that $f = hrh^{-1}$.*