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## CYCLIC DIFFERENCE SETS WITH PARAMETERS (511, 255, 127)

by Roland BACHER

This note describes the (equivalence classes of) cyclic difference sets with parameters (511, 255, 127). There are five non-isomorphic such classes which are listed at the end of this note. We give also a version of their triple intersection polynomial (our version differs by a numerical factor from the polynomial used in [EK]). The first and second belong to known families and are obtained by a geometric construction and by the GMW-construction described (for example) in [EK]. The last three appear to be exotic.

The technique used to produce all cyclic difference set with the above parameters is the same as in [C]. It uses heavy computer calculations and we will only sketch it.

Let  $v, k, \lambda$  be positive integers. Denote by  $C_v$  the cyclic group of order  $v$  which we identify with the additive group of the ring  $\mathbf{Z}/v\mathbf{Z}$ . A *cyclic difference set with parameters*  $(v, k, \lambda)$  is a subset  $D$  of cardinality  $k$  in  $C_v$  such that every element  $c \neq 0$  in  $C_v$  can be written in exactly  $\lambda$  different ways as

$$c = d_1 - d_2$$

with  $d_1, d_2$  in  $D$ . We identify  $D$  with its characteristic function. Hence  $D$  can be considered as an element of the group ring  $\mathbf{Z}C_v$ . Denote by  $\sigma : C_v \rightarrow C_v$  the automorphism of  $C_v$  which sends  $c \in C_v$  to its inverse  $-c$ . For an element  $X$  of  $\mathbf{Z}C_v$  set  $\bar{X} = \sigma(X)$ . An element  $X \in \mathbf{Z}C_v$  is a cyclic difference set if and only if all coefficients of  $X$  belong to the set  $\{0, 1\}$  and  $X$  satisfies the equation

$$X\bar{X} = \lambda C_v + (k - \lambda)\{0\}$$

for appropriate  $\lambda$  and  $k$ . Difference sets are preserved under translation in  $C_v$  and under automorphisms of  $C_v$ . Two difference sets related by a translation and an automorphism are considered equivalent.

For  $l$  prime to  $v$  we define an  $l$ -orbit  $X$  in  $C_v$  as an orbit of the automorphism  $\varphi_l: C_v \rightarrow C_v$  which sends  $c \in C_v$  to  $lc$ . The multiplier theorem (see e.g. section 1 in [EK]) shows that each cyclic difference set  $D'$  with parameters (511, 255, 127) is equivalent to a difference set  $D$  fixed under  $\varphi_2$ , i.e.  $D$  is a union of 2-orbits in  $C_{511}$ . We can hence restrict the search to such difference sets.

Reduction modulo 73 shows that all 2-orbits of  $C_{511}$  have 9 elements, with 3 exceptions:

$$\{0\}, \quad A = \{219, 365, 438\}, \quad B = \{73, 146, 292\}.$$

Since we are searching for a difference set  $D$  of cardinality 255, we see that  $\{0\}$  cannot be in  $D$  and that exactly one of  $A, B$  is in  $D$ . The automorphism  $\varphi_3$  of  $C_{511}$  exchanges the sets  $A$  and  $B$ , hence we may suppose that  $A \subset D$  and  $B \not\subset D$ .

In order to reduce the search further we use the fact that  $511 = 7 \cdot 73$  is not a prime. Let us introduce the obvious surjective group homomorphisms  $\pi_7: C_{511} \rightarrow C_7$  and  $\pi_{73}: C_{511} \rightarrow C_{73}$ . For  $M \in \mathbf{Z}C_{511}$  we set  $M_7 = \pi_7(M)$ ,  $\overline{M}_7 = \pi_7(\overline{M})$ ,  $M_{73} = \pi_{73}(M)$  and  $\overline{M}_{73} = \pi_{73}(\overline{M})$ . If the coefficients of  $M \in \mathbf{Z}C_v$  are constant on 2-orbits then so are the coefficients of  $M_l, \overline{M}_l$  for  $l = 7, 73$ . We take now for  $M$  a cyclic difference set  $D$  preserved by  $\varphi_2$  and satisfying  $A \subset D$ . Clearly  $D \in \mathbf{Z}C_{511}$  is an element with all coefficients constant on 2-orbits and in the set  $\{0, 1\}$ .

By considering the 2-orbits of  $C_7$  we have  $C_7 = X_0 \cup X_1 \cup X_2$  where  $X_0 = \{0\}$ ,  $X_1 = \{1, 2, 4\}$ ,  $X_2 = \{3, 5, 6\}$ . Hence we can write  $D_7 = \sum x_i X_i$ . All 2-orbits of  $D$  except  $A$  contain 9 elements. Since 2 is a square (mod 7), any 2-orbit of  $D$  distinct from  $A$  contributes 9 to  $x_0$  if its elements are in the kernel of the projection  $\pi_7$  or 3 to  $x_1$  or  $x_2$  according to whether it consists of squares or non-squares (mod 7). Moreover  $A \in D$  contributes 1 to  $x_1$ . All coefficients of  $D$  are either 0 or 1 and their sum is equal to 255, hence  $x_0, x_1, x_2$  are positive integers not greater than 73 such that  $x_0 \equiv 0 \pmod{9}$ ,  $x_1 \equiv 1 \pmod{3}$ ,  $x_2 \equiv 0 \pmod{3}$  and  $x_0 + 3x_1 + 3x_2 = 255$ . Since the difference set  $D$  satisfies the equation

$$D\overline{D} = 127C_v + 128\{0\}$$

in  $\mathbf{Z}C_{511}$  the projection  $D_7$  satisfies

$$D_7\overline{D}_7 = 128\{0\} + 127 \cdot 73C_7 = 9399\{0\} + 9271(X_1 \cup X_2).$$

The only solutions in  $\mathbf{Z}C_7$  satisfying all the above requirements are

$$x_0 = 27, \quad x_1 = 37, \quad x_2 = 39$$

$$x_0 = 45, \quad x_1 = 37, \quad x_2 = 33.$$

For  $C_{73}$  we have  $C_{73} = \cup_{j=0}^8 Y_j$  where  $Y_0 = \{0\}$  and

$$\begin{aligned} Y_1 &= \{1, 2, 4, 8, 16, 32, 64, 55, 37\}, \\ Y_2 &= \{3, 6, 12, 24, 48, 23, 46, 19, 38\}, \\ Y_3 &= \{5, 10, 20, 40, 7, 14, 28, 56, 39\}, \\ Y_4 &= \{9, 18, 36, 72, 71, 69, 65, 57, 41\}, \\ Y_5 &= \{11, 22, 44, 15, 30, 60, 47, 21, 42\}, \\ Y_6 &= \{13, 26, 52, 31, 62, 51, 29, 58, 43\}, \\ Y_7 &= \{17, 34, 68, 63, 53, 33, 66, 59, 45\}, \\ Y_8 &= \{25, 50, 27, 54, 35, 70, 67, 61, 49\}, \end{aligned}$$

is the decomposition of  $C_{73}$  into 2-orbits. We are searching for positive integers  $y_0, \dots, y_8$  not greater than 7 such that  $D_{73} = \sum_{j=0}^8 y_j Y_j$ . We already know that  $D$  contains a unique 2-orbit of cardinality 3 and that  $D$  does not contain 0. This implies that  $y_0 = 3$ . Moreover we have  $y_0 + 9 \sum_{j=1}^8 y_j = 255$  and  $D_{73} = \sum y_i Y_i$  satisfies

$$D_{73} \overline{D_{73}} = 128\{0\} + 127 \cdot 7C_{73} = 1017\{0\} + 889(\cup_{i=1}^8 Y_i).$$

A small computer program gives (up to multiplication by an invertible element in  $C_{73}$ ) the following four solutions (S1), ..., (S4) with  $y_0 = 3$ , and  $(y_1, \dots, y_8) =$

$$\begin{aligned} (S1) & \quad (7, 3, 3, 3, 3, 3, 3, 3), \\ (S2) & \quad (1, 5, 3, 3, 3, 3, 5, 5), \\ (S3) & \quad (1, 2, 4, 4, 5, 3, 4, 5), \\ (S4) & \quad (6, 3, 4, 3, 2, 2, 3, 5). \end{aligned}$$

Hence every cyclic difference set with parameters (511, 255, 127) is equivalent to a difference set  $D$  which projects onto one of the above 2 solutions in  $C_7$  and onto one of the above 4 solutions in  $C_{73}$ . One can reduce the number of cases somewhat more by considering the orbit of a difference set  $D$  under multiplication by 74 which is an invertible square (mod 7) (and hence keeps  $A$  fixed) and the identity (mod 73). There remain more than  $10^{10}$  unions of 2-orbits which have these projections. A computer program written in FORTRAN running about 50 hours on a SUN-workstation found (up to isomorphism) exactly five difference sets denoted by  $D_1, \dots, D_5$ . The notation  $D_i = (a_1, \dots, a_i)$  means that the difference set  $D_i$  is the union of

the 2-orbits generated by  $a_1, \dots, a_l$ . The polynomial  $P_i(x)$  associated to  $D_i$  is defined by

$$P_i(x) = \sum_{a, b \in C_{511}, 1 \leq a < b < 511} x^{\# \{D_i \cap (D_i + a) \cap (D_i + b)\}}$$

where  $D_i + a$  denotes the translate of  $D_i$  by  $a$ .

One easily checks that the usual triple intersection polynomial is just a scalar multiple of the one above.

Remark: For a difference set this polynomial does not depend on the choice of the fixed first block. For a symmetric design whose automorphism group is not transitive on the blocks it does in general. So this gives a condition fulfilled by a symmetric design coming from a difference set.

Two difference sets project onto (S1): These two correspond to known constructions and give factorizations of the elements  $D'_i(y)$  in the ring  $\mathbf{Z}[y]/(y^{511} - 1)$  where

$$D'_i(y) = \sum_{k \in D_i} y^k.$$

We also give a factorization of  $D'_i(y)$  in  $\mathbf{Z}[y]/(y^{511} - 1)$  (the factorization is not unique).

The first such difference set is

$$D_1 = (1, 7, 13, 17, 21, 23, 31, 35, 37, 39, 51, 53, 55, 59, 61, 75, \\ 77, 79, 83, 85, 91, 95, 103, 109, 123, 183, 187, 219, 223),$$

$$P_1(x) = 129540x^{63} + 255x^{127},$$

$D'_1(y)$  factorizes in  $\mathbf{Z}[y]/(y^{511} - 1)$  as

$$D'_1(y) = (1 + y^{73} + y^{146} + y^{292}) \\ (1 + y^{27} + y^{43} + y^{47} + y^{54} + y^{86} + y^{87} + y^{94} + y^{107} + y^{108} + y^{119} \\ + y^{172} + y^{174} + y^{177} + y^{179} + y^{185} + y^{188} + y^{195} + y^{197} + y^{205} \\ + y^{214} + y^{216} + y^{229} + y^{231} + y^{238} + y^{241} + y^{255} + y^{269} + y^{277} \\ + y^{279} + y^{299} + y^{309} + y^{315} + y^{344} + y^{345} + y^{348} + y^{353} + y^{354} \\ + y^{358} + y^{370} + y^{371} + y^{376} + y^{383} + y^{390} + y^{394} + y^{395} + y^{405} \\ + y^{410} + y^{413} + y^{428} + y^{432} + y^{441} + y^{447} + y^{453} + y^{458} + y^{462} \\ + y^{476} + y^{479} + y^{482} + y^{495} + y^{503} + y^{507} + y^{509} + y^{510}).$$

The second such difference set is

$$D_2 = (1, 15, 23, 27, 37, 45, 47, 51, 53, 55, 57, 61, 63, 75, 77, 79, 83, 85, \\ 87, 93, 103, 125, 127, 175, 183, 187, 219, 223, 255),$$

$$P_2(x) = 4536x^{55} + 121224x^{63} + 1512x^{71} + 2520x^{79} + 3x^{127},$$

$D'_2(y)$  factorizes in  $\mathbf{Z}[y]/(y^{511} - 1)$  as

$$D'_2(y) = (1 + y^{73} + y^{146} + y^{292}) \\ (1 + y^9 + y^{11} + y^{18} + y^{22} + y^{25} + y^{31} + y^{36} + y^{44} + y^{50} + y^{59} \\ + y^{62} + y^{65} + y^{67} + y^{72} + y^{88} + y^{100} + y^{118} + y^{119} + y^{124} + y^{130} \\ + y^{134} + y^{144} + y^{176} + y^{193} + y^{199} + y^{200} + y^{231} + y^{236} + y^{238} \\ + y^{239} + y^{247} + y^{248} + y^{260} + y^{261} + y^{268} + y^{271} + y^{285} + y^{288} \\ + y^{289} + y^{315} + y^{352} + y^{355} + y^{371} + y^{375} + y^{379} + y^{386} + y^{391} \\ + y^{398} + y^{400} + y^{413} + y^{433} + y^{441} + y^{443} + y^{445} + y^{451} + y^{462} \\ + y^{472} + y^{476} + y^{477} + y^{478} + y^{481} + y^{494} + y^{496}).$$

Two difference sets project onto (S2):

$$D_3 = (3, 5, 7, 11, 13, 25, 27, 31, 35, 39, 47, 51, 53, 59, 61, 63, 79, 85, \\ 91, 95, 107, 109, 111, 119, 127, 187, 191, 219, 223),$$

$$P_3(x) = 90x^{54} + 216x^{55} + 783x^{56} + 1746x^{57} + 3546x^{58} + 6912x^{59} \\ + 10692x^{60} + 12906x^{61} + 16461x^{62} + 17703x^{63} \\ + 17334x^{64} + 15615x^{65} + 10773x^{66} + 7236x^{67} + 4320x^{68} \\ + 1971x^{69} + 1026x^{70} + 270x^{71} + 57x^{72} + 84x^{73} + 54x^{74}$$

and

$$D_4 = (9, 17, 21, 23, 25, 27, 35, 39, 43, 53, 59, 63, 75, 79, 85, 87, 93, \\ 95, 107, 111, 117, 119, 125, 127, 171, 175, 187, 219, 255),$$

$$P_4(x) = 1353x^{55} + 25128x^{59} + 72813x^{63} + 29088x^{67} + 1413x^{71}.$$

No difference set projects onto (S3).

One difference set projects onto (S4):

$$D_5 = (1, 7, 9, 11, 25, 27, 31, 35, 37, 39, 47, 53, 55, 61, 77, 83, 85, \\ 87, 107, 109, 111, 119, 171, 175, 183, 187, 191, 219, 223),$$

$$P_5(x) = 27x^{53} + 246x^{54} + 135x^{55} + 900x^{56} + 1863x^{57} + 3276x^{58} + 6237x^{59} \\ + 9648x^{60} + 13851x^{61} + 17865x^{62} + 18486x^{63} + 16713x^{64} \\ + 14364x^{65} + 10458x^{66} + 8262x^{67} + 3834x^{68} + 2007x^{69} + 1134x^{70} \\ + 270x^{71} + 162x^{72} + 57x^{73}.$$

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