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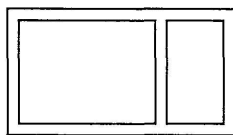
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For example, all elements of weight 3 in  $K_{m,n}$  can be represented (up to proper size and location) by the following picture:



or its vertical analogue. This picture represents a codeword of the form

$$E(k, l; r_1, r_2) + E(k, l; r_1, r_3) + E(k, l; r_2, r_3).$$

Thus, the number of codewords of weight 3 in  $K_{m,n}$  is equal to

$$w_3(K_{m,n}) = \binom{m}{2} \binom{n}{3} + \binom{m}{3} \binom{n}{2}.$$

Similarly, one can show that

$$w_4(K_{m,n}) = 3 \binom{m}{2} \binom{n}{4} + 9 \binom{m}{3} \binom{n}{3} + 3 \binom{m}{4} \binom{n}{2};$$

$$w_5(K_{m,n}) = 12 \binom{m}{2} \binom{n}{5} + 72 \binom{m}{3} \binom{n}{4} + 72 \binom{m}{4} \binom{n}{3} + 12 \binom{m}{2} \binom{n}{5} + 9 \binom{m}{3} \binom{n}{3}.$$

As a last remark, note that an upper bound for the weights in the associated code  $L_f$  is given by  $\frac{1}{8}n^3(n-1)$ , and that this bound is actually attained for some  $n$  if and only if there exists a Hadamard matrix of order  $n$ . This follows from, say, Corollary 3.

## 6. THE NUMBER OF PROPER 4-COLORINGS OF A GRAPH

Let  $G = (V, E)$  be a simple graph (no loops, no multiple edges) with vertex set  $V$  and edge set  $E$ . We will identify  $V$  with  $\{1, \dots, n\}$ , and denote the cardinality of  $E$  by  $e$ .

A 4-coloring of  $G$  is the assignment to every vertex of one among four fixed colors; such a coloring is *proper* if the colors assigned to the end vertices of any edge are distinct. For a survey on the 4-colorings of planar graphs, see [SK].

We will count the number of proper 4-colorings of  $G$ , in terms of the weight enumerator of a certain code of length  $3e$ .

STEP 1. *The defining equations for proper 4-colorings.*

As our palette of colors, we will choose the 4-set  $\{1, -1\}^2$ . The space of all 4-colorings of  $G$  can thus be identified with  $\{1, -1\}^{2n}$ , for example as follows:

*Convention.* If  $p = (p_1, \dots, p_{2n}) \in \{1, -1\}^{2n}$ , then the pair  $(p_i, p_{i+n})$  represents the color assigned to vertex  $i$ , for  $i = 1, \dots, n$ .

Consider now  $2n$  variables  $x_1, \dots, x_{2n}$ , and define

$$g_{i,j} := (1 + x_i x_j) (1 + x_{i+n} x_{j+n})$$

for  $1 \leq i, j \leq n$ .

If  $p$  is a 4-coloring of  $G$ , then  $g_{i,j}(p) = 0$  if and only if the colors assigned to vertices  $i$  and  $j$  are distinct.

Thus, a 4-coloring  $p$  of  $G$  is proper if and only if

$$g_{i,j}(p) = 0 \quad \text{for all } (i,j) \in E .$$

STEP 2. *Reduction to a single equation.*

Let

$$g := \sum_{(i,j) \in E} g_{i,j}^2 .$$

By construction,  $g$  satisfies the following properties:

- (1)  $g(p) \geq 0$  for all 4-colorings  $p$ ;
- (2)  $g(p) = 0$  if and only if  $p$  is proper.

Developing the expression for  $g$ , we obtain:

$$\begin{aligned} g &= \sum_E g_{i,j}^2 \\ &= \sum_E (1 + x_i x_j + x_{i+n} x_{j+n} + x_i x_j x_{i+n} x_{j+n})^2 \\ &= 4 \sum_E (1 + x_i x_j + x_{i+n} x_{j+n} + x_i x_j x_{i+n} x_{j+n}) \\ &= 4e + 4f, \end{aligned}$$

where

$$f := \sum_E (x_i x_j + x_{i+n} x_{j+n} + x_i x_j x_{i+n} x_{j+n}) .$$

(Here again, the computation was performed modulo  $x_i^2 = 1$  for all  $i$ .)

Obviously,  $f$  satisfies the following properties:

- (1)  $f(p) \geq -e$  for all 4-colorings  $p$ ;
- (2)  $f(p) = -e$  if and only if  $p$  is proper.

STEP 3. *The code associated with  $f$ .*

Let  $K_G := L_f^\perp$  be the dual of the code  $L_f$  associated with  $f$ . To describe it, we consider the map

$$\begin{aligned}\phi_G: \mathbf{F}_2^{3e} &\rightarrow \mathbf{F}_2^{2n} \\ E_{i,j} &\mapsto e_i + e_j \\ E_{i+n,j+n} &\mapsto e_{i+n} + e_{j+n} \\ E_{i+2n,j+2n} &\mapsto e_i + e_j + e_{i+n} + e_{j+n} \quad ((i,j) \in E)\end{aligned}$$

where  $\{E_{i,j}, E_{i+n,j+n}, E_{i+2n,j+2n}\}_{(i,j) \in E}$  and  $\{e_i\}_{i \in V}$  denote the standard bases of the left and right spaces, respectively. By construction,  $K_G = \text{Ker}(\phi_G)$ .

Here again, as a direct consequence of Theorem 4 and of the stated properties of  $f$ , we have:

**THEOREM 7.** *Let  $K_G$  be the code of length  $3e$  defined above, and let  $P_G(T)$  denote its weight enumerator. Then, the number  $\chi_G(4)$  of proper 4-colorings of  $G$ , is given by*

$$\chi_G(4) = \frac{1}{4^{e-n} e!} P_G^{(e)}(-1),$$

where  $P_G^{(e)}(-1)$  denotes the value at  $-1$  of the  $e$ -th derivative of  $P_G(T)$ .

*Proof.* Apply the formula of Theorem 4 by replacing

- $N$ , the length of the code, by  $3e$ ;
- $\nu$ , a lower bound for the range of  $f$ , by  $-e$ ;
- $n$ , the number of variables in  $f$ , by  $2n$ . □

Note that there are some obvious elements of weight 3 in  $K_G$ , namely

$$E_{i,j} + E_{i+n,j+n} + E_{i+2n,j+2n} \quad ((i,j) \in E).$$

More interestingly, every cycle of length  $r$  in  $G$  gives rise to at least 3 elements of weight  $r$  in  $K_G$ . Indeed, if  $(i_1, \dots, i_r)$  is a cycle, then

$$E_{i_1+kn, i_2+kn} + E_{i_2+kn, i_3+kn} + \dots + E_{i_r+kn, i_1+kn} \in K_G,$$

for  $k = 0, 1, 2$ .

One can go a little bit further. A somewhat technical computation shows that  $P_G(T)$  can be decomposed as follows:

$$P_G(T) = (1 + T^3)^e P_{\bar{K}} \left( \frac{T}{1 - T + T^2} \right),$$

where  $\tilde{K}$  is the  $\mathbf{F}_4$ -code defined by the exact sequence

$$0 \rightarrow \tilde{K} \rightarrow \mathbf{F}_4^e \rightarrow \mathbf{F}_4^n,$$

with weight enumerator  $P_{\tilde{K}}(T)$ . (The map on the right sends the basis vector corresponding to an edge, to the formal sum of its two endvertices.)

With the above formula, we find that  $P_G^{(e)}(-1) = 3^e e! P_{\tilde{K}}\left(-\frac{1}{3}\right)$ . Plugging this into Theorem 7, we obtain

$$\chi_G(4) = \frac{3^e}{4^{e-n}} P_{\tilde{K}}\left(-\frac{1}{3}\right).$$

In particular,  $G$  is 4-colorable if and only if  $P_{\tilde{K}}\left(-\frac{1}{3}\right) \neq 0$ . See also [E, Theorem 5.7], for a different formulation and proof of this formula.

#### REFERENCES

- [B] BARKER, R. H. Group synchronizing of binary digital systems. In *Communication Theory*, W. Jackson, Ed., London: Butterworth, 1953, 273-287.
- [E] ELIAHOU, S. An algebraic criterion for a graph to be four-colourable. *Aportaciones Matemáticas, Notas de Investigación 6* (1992), 3-27.
- [EKS] ELIAHOU, S., M. KERVAIRE and B. SAFFARI. On Golay polynomial pairs. *Adv. Appl. Math.* 12 (1991), 235-292.
- [G] GOLAY, M. J. E. Static multislit spectrometry and its application to the panoramic display of infrared spectra. *J. Opt. Soc. Amer.* 41 (1951), 468-472.
- [H] HADAMARD, J. Résolution d'une question relative aux déterminants. *Bull. Sci. Math. (2)* 17 (1893), 240-246.
- [M] MACWILLIAMS, F. J. A theorem on the distribution of weights in a systematic code. *Bell Syst. Tech. J.* 42 (1963), 79-94.
- [MS] MACWILLIAMS, F. J. and N. J. A. SLOANE. *The Theory of Error-Correcting Codes*. North-Holland, 1977.
- [SK] SAATY, T. L. and P. C. KAINEN. *The Four-Color Problem, Assaults and Conquest*. McGraw-Hill, New York, 1977.
- [SY] SEBERRY, J. and M. YAMADA. Hadamard matrices, Sequences, and Block Designs. In *Contemporary Design Theory: A Collection of Surveys*, J. H. Dinitz and D. R. Stinson, Eds., Wiley-Interscience, New York, 1992, 431-560.

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