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and therefore

$$\alpha_{\gamma} = \frac{1}{2^{N-\alpha-\dim L} \alpha!} P_{K}^{(\alpha)}(-1) .$$

Multiplying both sides by $2^{n-\dim L}$, and plugging in equation (1), we obtain the claimed formula for $|f^{-1}(v)|$.

COROLLARY 5. Let v_{min} be the least value assumed by f on binary points. Then

$$\frac{1}{2}(N+v_{min}) = the order of -1 as a root of P_K(T).$$

5. The number of Hadamard matrices of order n

A Hadamard matrix is a square matrix H of order n with entries in $\{+1, -1\}$, satisfying the relation

$$H \cdot H' = nI_n$$

 $(H^{\top}$ denotes the transpose of H, and I_n the identity matrix of order n.)

It is well known that the order of a Hadamard matrix can only be 1, 2 or a multiple of 4. Conversely, the existence of a Hadamard matrix of order nfor every $n \equiv 0 \mod 4$ is a longstanding conjecture, due to Jacques Hadamard [H]. The smallest open case currently occurs at n = 428. For a survey on Hadamard matrices, see [SY].

The theory exposed above yields a counting formula for Hadamard matrices of order *n*, in terms of the weight enumerator of a certain binary linear code of length $\binom{n}{2}^2$.

STEP 1. Defining equations for Hadamard matrices.

We represent binary matrices of order *n* as points $p = (p_{i,j}) \in \{1, -1\}^{n^2}$. Considering n^2 variables $\{x_{i,j}\}_{1 \le i,j \le n}$, let

$$g_{k,l} = \sum_{r=1}^{n} x_{k,r} x_{l,r}$$

If $p = (p_{i,j})$ is a binary matrix, then $g_{k,l}(p)$ is the dot product of the k-th and l-th rows of p. Thus, a binary matrix p is Hadamard if and only if

$$g_{k,l}(p) = 0$$
 for all $1 \leq k < l \leq n$.

STEP 2. *Reduction to a single equation*. Let

$$g = \sum_{1 \leq k < l \leq n} g_{k,l}^2.$$

By construction, we have the following properties:

- (1) $g(p) \ge 0$ for every binary matrix p;
- (2) g(p) = 0 if and only if p is Hadamard.

Developing the expression for g, we obtain:

$$g = \sum_{k < l} g_{k,l}^{2}$$

= $\sum_{k < l} (\sum_{r} x_{k,r} x_{l,r})^{2}$
= $\sum_{k < l} (n + 2 \sum_{r < s} x_{k,r} x_{l,r} x_{k,s} x_{l,s})$
= $n {n \choose 2} + 2f$,

where

$$f := \sum_{k < l} \sum_{r < s} x_{k,r} x_{l,r} x_{k,s} x_{l,s} .$$

(Of course, the above computation is performed modulo the relations $x_{i,j}^2 = 1$ for all i, j.)

The following properties of $f = \frac{1}{2} \left(g - n {n \choose 2} \right)$ derive instantly from those of g:

- (1) $f(p) \ge -\frac{1}{2}n\binom{n}{2}$ for every binary matrix p;
- (2) $f(p) = -\frac{1}{2}n\binom{n}{2}$ if and only if p is Hadamard.

STEP 3. The code associated with f.

Let $K_n := L_f^{\perp}$ denote the dual of the binary code L_f associated with f, as defined in Section 3. Explicitly, we consider the map

$$\phi_n: \qquad \mathbf{F}_2^{\binom{n}{2}^2} \rightarrow \mathbf{F}_2^{n^2} \\ E(k, l; r, s) \qquad \mapsto \qquad e_{k, r} + e_{l, r} + e_{k, s} + e_{l, s} ,$$

where $\{E(k, l; r, s)\}_{1 \le k < l \le n, 1 \le r < s \le n}$ and $\{e_{i, j}\}_{1 \le i, j \le n}$ denote the standard bases of the left and right spaces, respectively; by construction then, $K_n = \text{Ker}(\phi_n)$.

As a direct consequence of Theorem 4 and of the above-mentioned properties of f, we obtain the

THEOREM 6. Let K_n (*n* even) be the code of length $\binom{n}{2}^2$ defined as the kernel of the above map $\phi_n : \mathbf{F}_2^{\binom{n}{2}^2} \to \mathbf{F}_2^{n^2}$. Let $P_n(T)$ denote the weight enumerator of K_n . Then the number h(n) of Hadamard matrices of order n is given by

$$h(n) = \frac{1}{2^{\beta(n)} \alpha(n)!} \cdot P_n^{(\alpha(n))}(-1),$$

where

- 1. $\alpha(n) = n^2(n-1)(n-2)/8;$
- 2. $\beta(n) = n^3(n-1)/8 n^2;$
- 3. $P_n^{(\alpha(n))}(-1)$ denotes the $\alpha(n)$ -th derivative of $P_n(T)$, evaluated at -1.

Proof. In the formula of Theorem 4, replace:

- N, the length of the code, by $\binom{n}{2}^2$;
- v, a lower bound for the values of f, by $-\frac{1}{2}n\binom{n}{2}$; and
- *n*, the number of variables in f, by n^2 .

Thus, the determination of the weight enumerator of K_n is an important problem. We will give below, without proof, the number of codewords of weight 3, 4 and 5 of K_n . (Of course, there are no words of weight 1 or 2 in K_n .) But the problem can be generalized a little bit, as follows. Consider the map

$$\phi_{m,n}: \mathbf{F}_{2}^{\binom{m}{2}\binom{n}{2}} \rightarrow \mathbf{F}_{2}^{mn}$$

$$E(k,l;r,s) \mapsto e_{k,r} + e_{l,r} + e_{k,s} + e_{l,s},$$

where now, the indices k < l range from 1 to *m* instead of 1 to *n*. We denote by $K_{m,n}$ the kernel of $\phi_{m,n}$.

Let $\Gamma = \{1, ..., m\} \times \{1, ..., n\}$. We can think of the vector basis $e_{i, j}$ as the point on row *i* and column *j* in the grid Γ , and of E(k, l; r, s) as the rectangle determined by rows k, l and columns r, s in Γ . The image of E(k, l; r, s) under $\phi_{m,n}$, then, is the formal sum of its four corners.

Thus, an element of weight w in $K_{m,n}$ can be pictured as a set of w rectangles in the grid Γ , such that every point in the grid coincides with an *even* number of corners of the rectangles in the set.

For example, all elements of weight 3 in $K_{m,n}$ can be represented (up to proper size and location) by the following picture:



or its vertical analogue. This picture represents a codeword of the form

$$E(k, l; r_1, r_2) + E(k, l; r_1, r_3) + E(k, l; r_2, r_3)$$
.

Thus, the number of codewords of weight 3 in $K_{m,n}$ is equal to

$$w_{3}(K_{m,n}) = \binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2}$$

Similarly, one can show that

$$w_{4}(K_{m,n}) = 3 \binom{m}{2} \binom{n}{4} + 9 \binom{m}{3} \binom{n}{3} + 3 \binom{m}{4} \binom{n}{2};$$

$$w_{5}(K_{m,n}) = 12 \binom{m}{2} \binom{n}{5} + 72 \binom{m}{3} \binom{n}{4} + 72 \binom{m}{4} \binom{n}{3} + 12 \binom{m}{2} \binom{n}{5} + 9 \binom{m}{3} \binom{n}{3}$$

As a last remark, note that an upper bound for the weights in the associated code L_f is given by $\frac{1}{8}n^3(n-1)$, and that this bound is actually attained for some *n* if and only if there exists a Hadamard matrix of order *n*. This follows from, say, Corollary 3.

6. The number of proper 4-colorings of a graph

Let G = (V, E) be a simple graph (no loops, no multiple edges) with vertex set V and edge set E. We will identify V with $\{1, ..., n\}$, and denote the cardinality of E by e.

A 4-coloring of G is the assignment to every vertex of one among four fixed colors; such a coloring is *proper* if the colors assigned to the end vertices of any edge are distinct. For a survey on the 4-colorings of planar graphs, see [SK].

We will count the number of proper 4-colorings of G, in terms of the weight enumerator of a certain code of length 3e.

STEP 1. The defining equations for proper 4-colorings.

As our palette of colors, we will choose the 4-set $\{1, -1\}^2$. The space of all 4-colorings of G can thus be identified with $\{1, -1\}^{2n}$, for example as follows: