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4. ON THE LEAST VALUE OF  $f$ 

Let us consider again an integral polynomial  $f = u_1 + \cdots + u_N$ , with  $u_i \in M_n$  for all  $i$ . By Theorem 2 and the MacWilliams identity, the cardinality of  $f^{-1}(v)$  can be expressed in terms of the weight enumerator of the dual code  $K_f = L_f^\perp$ , for every  $v$  in  $\mathbf{Z}$ .

In this section, we will obtain another such formula for  $|f^{-1}(v)|$ , provided  $v$  is a lower bound for the range of  $f$ . These results could be applied to "count" the number of binary zeros of  $f$ , since  $v = 0$  is a lower bound for the range of  $f^2$ , and  $f^2$  has as many binary zeros as  $f$  does.

**THEOREM 4.** *Let  $f = u_1 + \cdots + u_N$  with  $u_i \in M_n$  for all  $i$ , and let  $K := L_f^\perp$  be the dual of the code  $L_f$  associated with  $f$ , with weight enumerator  $P_K(T)$ . Assume that  $v \in \mathbf{Z}$ ,  $v \equiv N \pmod{2}$ , is a lower bound for  $f$ , i.e.*

$$v \leq f(p)$$

for all  $p \in \{1, -1\}^n$ . Then we have

$$|f^{-1}(v)| = \frac{1}{2^\beta \cdot \alpha!} \cdot P_K^{(\alpha)}(-1)$$

where

1.  $\alpha = \alpha(v) = (N + v)/2$ ,
2.  $\beta = \beta(v) = (N - v)/2 - n$ , and
3.  $P_K^{(\alpha)}(-1)$  denotes the value at  $-1$  of the  $\alpha$ -th derivative of  $P_K(T)$ .

*Proof.* Let  $P_L(T) = \sum_{i=0}^N a_i T^i$  denote the weight enumerator of  $L = L_f$ , and let  $\gamma = \gamma(v) = (N - v)/2$ . By Corollary 3, we have  $\deg P_L \leq \gamma$  since  $f(p) \geq v$  for all  $p$ , and

$$(1) \quad |f^{-1}(v)| = 2^{n - \dim L} \cdot \alpha_\gamma.$$

Now, by the MacWilliams identity, the weight enumerator of  $K$  is given by

$$\begin{aligned} P_K(T) &= \frac{1}{|L|} \cdot \sum_{i=0}^{\gamma} a_i (1+T)^{N-i} (1-T)^i \\ &= \frac{1}{|L|} \cdot (1+T)^{N-\gamma} \cdot (\alpha_\gamma (1-T)^\gamma + (1+T)Q(T)), \end{aligned}$$

where  $Q(T)$  is some polynomial in  $T$ . Note that  $N - \gamma = \alpha = (N + v)/2$ .

To extract  $\alpha_\gamma$  from the above expression, we derive  $\alpha$  times, and evaluate at  $T = -1$ :

$$\begin{aligned} P_K^{(\alpha)}(-1) &= \frac{1}{|L|} \alpha! \alpha_\gamma 2^\gamma \\ &= \frac{1}{2^{\dim L}} \alpha! \alpha_\gamma 2^{N-\alpha}, \end{aligned}$$

and therefore

$$\alpha_\gamma = \frac{1}{2^{N-\alpha-\dim L} \alpha!} P_K^{(\alpha)}(-1).$$

Multiplying both sides by  $2^{n-\dim L}$ , and plugging in equation (1), we obtain the claimed formula for  $|f^{-1}(v)|$ .  $\square$

COROLLARY 5. *Let  $v_{min}$  be the least value assumed by  $f$  on binary points. Then*

$$\frac{1}{2} (N + v_{min}) = \text{the order of } -1 \text{ as a root of } P_K(T). \quad \square$$

### 5. THE NUMBER OF HADAMARD MATRICES OF ORDER $n$

A *Hadamard matrix* is a square matrix  $H$  of order  $n$  with entries in  $\{+1, -1\}$ , satisfying the relation

$$H \cdot H^T = nI_n.$$

( $H^T$  denotes the transpose of  $H$ , and  $I_n$  the identity matrix of order  $n$ .)

It is well known that the order of a Hadamard matrix can only be 1, 2 or a multiple of 4. Conversely, the existence of a Hadamard matrix of order  $n$  for every  $n \equiv 0 \pmod{4}$  is a longstanding conjecture, due to Jacques Hadamard [H]. The smallest open case currently occurs at  $n = 428$ . For a survey on Hadamard matrices, see [SY].

The theory exposed above yields a counting formula for Hadamard matrices of order  $n$ , in terms of the weight enumerator of a certain binary linear code of length  $\binom{n}{2}^2$ .

#### STEP 1. *Defining equations for Hadamard matrices.*

We represent binary matrices of order  $n$  as points  $p = (p_{i,j}) \in \{1, -1\}^{n^2}$ . Considering  $n^2$  variables  $\{x_{i,j}\}_{1 \leq i,j \leq n}$ , let

$$g_{k,l} = \sum_{r=1}^n x_{k,r} x_{l,r}.$$

If  $p = (p_{i,j})$  is a binary matrix, then  $g_{k,l}(p)$  is the dot product of the  $k$ -th and  $l$ -th rows of  $p$ . Thus, a binary matrix  $p$  is Hadamard if and only if

$$g_{k,l}(p) = 0 \quad \text{for all } 1 \leq k < l \leq n.$$