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# 10. Appendix

This appendix contains results due to Brian Bowditch, published here with his permission.

We recall that a finite-sided closed convex cell of  $\mathbf{H}^{n+1}$  is said to be pyramidal at an ideal point p if any two faces whose closures contain p meet in  $\mathbf{H}^{n+1}$ . The intersection of such a convex cell with a horosphere centred at p is a euclidean finite-sided closed convex cell of dimension n (provided the horosphere only meets faces which have p as an ideal point). One way to see this is to use the upper half-space model with p equal to the point at infinity. Conversely, given a convex finite-sided *n*-dimensional euclidean cell, we can think of this cell as lying in a horosphere which is a horizontal subspace in the upper half-space model. This gives rise to an (n + 1)-dimensional hyperbolic convex cell, by taking the intersection of vertical half-spaces determined by the half-spaces defining the euclidean convex cell. We use the names "pyramidal" and "non-pyramidal" for convex euclidean cells if the corresponding hyperbolic cells are pyramidal or non-pyramidal respectively. A euclidean convex cell is non-pyramidal if and only if it has disjoint faces. If a euclidean cell is pyramidal, then there is a face which is the intersection of all other faces, that is there is a unique minimal face. A pyramidal euclidean *n*-cell is the product of an *i*-dimensional cell with the cone on a spherical (n-i-1)-dimensional cell. (The cone point is placed at the centre of the (n - i - 1)-dimensional sphere.)

Let M be a connected euclidean similarity n-dimensional manifold which is the union of a locally finite set of closed subsets  $\{X_i\}$ . Each  $X_i$  has an induced similarity structure which is isomorphic to that of a closed finite-sided euclidean convex polyhedron. There are only a finite number of distinct similarity classes of  $X_i$ . The intersection of any face of any  $X_i$  with any face of any  $X_j$  is a common face of each. This implies that M has the structure of a locally finite polyhedral cell complex. Let G be a group of similarities of M which preserve the cell structure. Suppose that the number of orbits of non-pyramidal polyhedra is finite.

THEOREM 10.1 (Bowditch). Under the above assumptions, the number of orbits of cells is finite. Moreover, the number of orbits is bounded in terms of the number of orbits of non-pyramidal cells and the geometry (up to similarity) of the  $X_i$ .

Bowditch has suggested that if there is one or more pyramidal polyhedral cell, then one should be able to prove that G is a finite group. It would follow

that G consists of euclidean isometries and that M contains only a finite number of cells. This conjecture remains open.

*Proof of 10.1.* Let X be the union of the non-pyramidal cells in M, and let Y be the union of cells which meet X. Note that  $X \subset Y$ .

Now suppose there is a top-dimensional cell which is not in Y and let  $\sigma$  be its unique minimal face. Then  $\sigma$  is similar to  $\mathbb{R}^i$  for some *i*. If  $\alpha$  is any cell meeting  $\sigma$ , then  $\sigma \subset \alpha$  since  $\sigma$  is minimal. Clearly  $\alpha$  is not in X. Therefore  $\sigma$  is the unique minimal face of  $\alpha$ . We have seen above that  $\alpha$  is the product of  $\sigma$  and the cone on a convex subset  $\mathbb{S}^{n-i-1}$ . It follows that the union of the cells meeting  $\sigma$  is the product of  $\sigma$  with the cone on  $\mathbb{S}^{n-i-1}$ . It follows that the the cell structure of M is finite, G is a finite group and  $X = \emptyset$ . The other possibility is that Y = M.

Let  $K \,\subset X$  be a finite union of cells such that GK = X. The cell structure of M is locally finite, with a bound for the number of cells in any small neighbourhood being given by the geometry of the  $X_i$ . The number of cells of M which meet K is bounded by the number of cells of K and the maximum possible number of cells meeting a fixed small neighbourhood of any fixed point of K. This gives an upper bound for the number of orbits of cells of M under the action of G in case Y = M. If  $X = Y = \emptyset$ , then the number of cells of M is bounded by the geometry of the  $X_i$ .  $\Box$ 

We apply Theorem 10.1 to find out a litte more about the spaces that arise in Poincaré's Theorem. Suppose the hypotheses Pairing( $\mathcal{P}, R, A$ ), Connected( $\mathcal{P}, R$ ), Finite( $\mathcal{P}$ ) and Cyclic( $\mathcal{P}, R, A$ ) are satisfied for a set of convex cells (see Definition 2.8) in  $\mathbf{H}^n$ . To each convex cell we adjoin the ideal points, so as to obtain a compact space. The face-pairings are defined on the closures of the faces. Let  $\overline{Q}$  be the quotient of the disjoint union of the extended cells by the face-pairings, endowed with the quotient topology.

THEOREM 10.2.  $\overline{Q}$  is a compact hausdorff space.

**Proof of 10.2.** Let X be the disjoint union of the closures of the convex cells. So X is compact and hausdorff. We first show that the inverse image of a point under the quotient map  $X \rightarrow \overline{Q}$  is a finite set. This is clear from Theorem 4.13 for any point which is not an ideal point. For an ideal point p, we can construct a similarity manifold to which Theorem 10.1 applies, by developing a horosphere centred at p into  $\mathbb{R}^{n-1}$ . More details, which will help the interested reader with the construction of the similarity manifold, are given in the discussion of Definition 6.2.

A pyramidal cell in  $\mathbb{R}^{n-1}$  corresponds to a convex cell in  $\mathbb{H}^n$  together with an ideal point p in its boundary, such that any two faces with closures containing p meet inside  $\mathbb{H}^n$ . A non-pyramidal cell corresponds to a convex cell in  $\mathbb{H}^n$  and an ideal point p contained in the closures of two nonintersecting faces of the convex cell. The hypothesis needed in order to apply Theorem 10.1, that there are only a finite number of orbits of non-pyramidal cells, comes from the fact that there are only a finite number of pairs of faces and therefore only a finite number of pairs of non-intersecting faces which meet at infinity.

It follows that the inverse image in X of any point of Q is finite. Moreover the number of points in the inverse image is bounded by a fixed integer N. Two points  $x, y \in X$  are mapped to the same point of  $\overline{Q}$  if and only if there is a sequence  $(x_0, ..., x_n)$  such that  $x = x_0$ ,  $y = x_n$  and  $x_{i+1} = A(F_i)(x_i)$ , where  $x_i \in F_i$  and  $x_{i+1} \in R(F_i)$ . (Here (R, A) is the glueing data.) We may take  $n \leq N$ . It follows easily from compactness and the finiteness of the situation that the map  $X \to \overline{Q}$  is closed. Therefore  $\overline{Q}$  is hausdorff.  $\Box$ 

# REFERENCES

- [Ale54] ALEKSANDROV, A.D. Filling space by polyhedra. Vestnik. Leningrad Univ. Math. 2 (1954), 33-34.
- [Apa86] APANASOV, B.N. Filling a space by polyhedra and deformation of incomplete hyperbolic structures. Siberian Math. J. 27 (1986), 473-485. (English translation.)
- [Bea83] BEARDON, A.F. The Geometry of Discrete Groups. Springer-Verlag, 1983.
- [Bow93] BOWDITCH, B. Geometrical finiteness for hyperbolic groups. J. Funct. Anal. 113 (1993), 245-317.
- [BP92] BENEDETTI, R. and C. PETRONIO. Lectures on hyperbolic geometry. Springer-Verlag, 1992.
- [BSS89] BLUM, L., M. SHUB and S. SMALE. On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines. Bull. Amer. Math. Soc. 21 (1989), 1-46.
- [dlH91] DE LA HARPE, P. An invitation to Coxeter groups. In: Group theory from a geometrical viewpoint, Ghys-Haefliger-Verjovsky editors, World Scientific Publishers, Singapore, 1991.
- [dR71] DE RHAM, G. Sur les polygones générateurs de groupes fuchsiens. Enseign. Math. 17 (1971), 49-61.
- [Mas71] MASKIT, B. On Poincaré's theorem for fundamental polygons. Adv. Math. 7 (1971), 219-230.
- [Mas88] Kleinian Groups. Springer-Verlag, 1988.

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[Mor78] MOROKUMA, T. A characterization of fundamental domains of discontinuous groups acting on real hyperbolic spaces. J. Fac. Sci. Univ. Tokyo Section 1A Math. 25 (1978), 157-183.