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Artikel: UNIMODULAR LATTICES WITH A COMPLETE ROOT SYSTEM
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Thus,

$$f(u) = \prod_{i \in S} (1 + (q_i - 1)x_i) \cdot \prod_{i \in S'} (1 - x_i) ,$$

where $S \subset \{1, ..., s\}$ is the set of indices i for which $u_i = 0$, and $S' \subset \{1, ..., s\}$ the set of indices i for which $u_i \neq 0$.

Another way of writing $f(u)$ is

$$f(u) = \prod_{i=1}^{s} (1 - x_i)^{w(u_i)} \cdot (1 + (q_i - 1)x_i)^{1 - w(u_i)} .$$

Plugging this formula into $\sum_{u \in M} f(u)$, we get

$$\begin{aligned}\sum_{u \in M} f(u) &= \prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot \sum_{u \in M} \prod_{i=1}^{s} \left(\frac{1 - x_i}{1 + (q_i - 1)x_i}\right)^{w(u_i)} \\ &= \prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, ..., \frac{1 - x_s}{1 + (q_s - 1)x_s}\right) .\end{aligned}$$

Comparing the two expressions for $\sum_{u \in M} f(u)$, we get the theorem.

5. The deficiency

The main further necessary condition for a root system to be contained in an even unimodular lattice of the same rank is provided by the notion of deficiency (Defekt) introduced and studied in [KV].

If R is a root system of rank n, the *deficiency* of R, denoted $d(R)$, is the difference $n - m$, where m is the maximal cardinality of a set $\{a_1, ..., a_m\} \subset R$ of mutually orthogonal roots

$$(a_i, a_j) = 2\delta_{ij}, \qquad \text{for all } 1 \leqslant i, j \leqslant m .$$

We use this notion only if all roots in R have the same scalar square 2.

If $R = R_1 \boxplus R_2$, then $d(R) = d(R_1) + d(R_2)$. The values of the deficiency for the irreducible root systems are

$$d(\mathbf{A}_l) = \left[\tfrac{l}{2}\right] ,$$

$$d(\mathbf{D}_l) = \begin{cases} 0 & \text{for } l \text{ even,} \\ 1 & \text{for } l \text{ odd ,} \end{cases}$$

$$d(\mathbf{E}_6) = 2, \; d(\mathbf{E}_7) = d(\mathbf{E}_8) = 0 .$$

By Satz 5 of [KV], if R is the (complete) root system of an even unimodular lattice of rank 32, then

$$d(R) = 0, 8, 12, 14, 15 \text{ or } 16 .$$

The proof consists in constructing from the given lattice a new lattice L, still of rank 32 and containing the orthogonal sum of $m = 32 - d(R)$ copies of $\mathbf{Z}$. Thus, $L = \mathbf{Z}^m \boxplus L_0$, where L_0 is again unimodular and of rank $d(R)$. (Hence, $\operatorname{rank}(L_0) \leqslant 16$.)

By Martin Kneser's classification of unimodular (positive definite) lattices of rank $\leqslant 16$, the rank of L_0, i.e. $d(R)$ can only take the above values. (See [Kn], Satz 1.)

In setting up the tables we conveniently use the deficiency to discriminate the various root systems R according to the value of $d(R)$.

6. THE TABLES

We now proceed to list the *indecomposable* even unimodular lattices L of rank 32 with a complete root system R.

The presence in R of a factor of type $\mathbf{E}_8$ would produce a unimodular sublattice $\mathbf{ZE}_8 = L_0 \subset L$, and hence a decomposition $L = L_0 \boxplus L_1$ for some (even) unimodular L_1 of rank 24. Hence, we assume throughout that R has the form

$$R = \mathbf{A}_{l_1} \boxplus \ldots \boxplus \mathbf{A}_{l_r} \boxplus \mathbf{D}_{m_1} \boxplus \ldots \boxplus \mathbf{D}_{m_s} \boxplus \mathbf{mE}_6 \boxplus \mathbf{nE}_7 ,$$

with no factor of type $\mathbf{E}_8$.

Altogether there are $N = 88523$ such systems (of rank 32). The possible dimensions for $\mathbf{mE}_6 \boxplus \mathbf{nE}_7$ are

$$D = \{0, 6, 7, 12, 13, 14, 18, 19, 20, 21, 24, 25, 26, 27, 28, 30, 31, 32\}$$

and for $d \in D$, there is a unique pair (m, n) such that $d = 6m + 7n$. Hence

$$N = \textstyle\sum_{d \in D} \sum_{i=0}^{32-d} p(i) q(32 - d - i) ,$$

where $p(i)$ is the number of partitions of i and $q(j)$ is the number of partitions $(j_1, \ldots, j_t)$ of j with $4 \leqslant j_1 \leqslant \ldots \leqslant j_t$. (Of course, we use the convention $p(0) = q(0) = 1$.)

Among these, only 21209 have an acceptable deficiency, i.e. $d = 0, 8, 12, 14, 15$ or 16. They are distributed as follows: