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Thus,

$$f(u) = \prod_{i \in S} (1 + (q_i - 1)x_i) \cdot \prod_{i \in S'} (1 - x_i),$$

where  $S \subset \{1, \dots, s\}$  is the set of indices  $i$  for which  $u_i = 0$ , and  $S' \subset \{1, \dots, s\}$  the set of indices  $i$  for which  $u_i \neq 0$ .

Another way of writing  $f(u)$  is

$$f(u) = \prod_{i=1}^s (1 - x_i)^{w(u_i)} \cdot (1 + (q_i - 1)x_i)^{1 - w(u_i)}.$$

Plugging this formula into  $\sum_{u \in M} f(u)$ , we get

$$\begin{aligned} \sum_{u \in M} f(u) &= \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot \sum_{u \in M} \prod_{i=1}^s \left( \frac{1 - x_i}{1 + (q_i - 1)x_i} \right)^{w(u_i)} \\ &= \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot P_M \left( \frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_s}{1 + (q_s - 1)x_s} \right). \end{aligned}$$

Comparing the two expressions for  $\sum_{u \in M} f(u)$ , we get the theorem.

## 5. THE DEFICIENCY

The main further necessary condition for a root system to be contained in an even unimodular lattice of the same rank is provided by the notion of deficiency (Defekt) introduced and studied in [KV].

If  $R$  is a root system of rank  $n$ , the *deficiency* of  $R$ , denoted  $d(R)$ , is the difference  $n - m$ , where  $m$  is the maximal cardinality of a set  $\{a_1, \dots, a_m\} \subset R$  of mutually orthogonal roots

$$(a_i, a_j) = 2\delta_{ij}, \quad \text{for all } 1 \leq i, j \leq m.$$

We use this notion only if all roots in  $R$  have the same scalar square 2.

If  $R = R_1 \boxplus R_2$ , then  $d(R) = d(R_1) + d(R_2)$ . The values of the deficiency for the irreducible root systems are

$$\begin{aligned} d(\mathbf{A}_l) &= \left[ \frac{l}{2} \right], \\ d(\mathbf{D}_l) &= \begin{cases} 0 & \text{for } l \text{ even,} \\ 1 & \text{for } l \text{ odd,} \end{cases} \end{aligned}$$

$$d(\mathbf{E}_6) = 2, \quad d(\mathbf{E}_7) = d(\mathbf{E}_8) = 0.$$

By Satz 5 of [KV], if  $R$  is the (complete) root system of an even unimodular lattice of rank 32, then

$$d(R) = 0, 8, 12, 14, 15 \text{ or } 16 .$$

The proof consists in constructing from the given lattice a new lattice  $L$ , still of rank 32 and containing the orthogonal sum of  $m = 32 - d(R)$  copies of  $\mathbf{Z}$ . Thus,  $L = \mathbf{Z}^m \boxplus L_0$ , where  $L_0$  is again unimodular and of rank  $d(R)$ . (Hence,  $\text{rank}(L_0) \leq 16$ .)

By Martin Kneser's classification of unimodular (positive definite) lattices of rank  $\leq 16$ , the rank of  $L_0$ , i. e.  $d(R)$  can only take the above values. (See [Kn], Satz 1.)

In setting up the tables we conveniently use the deficiency to discriminate the various root systems  $R$  according to the value of  $d(R)$ .

## 6. THE TABLES

We now proceed to list the *indecomposable* even unimodular lattices  $L$  of rank 32 with a complete root system  $R$ .

The presence in  $R$  of a factor of type  $\mathbf{E}_8$  would produce a unimodular sublattice  $\mathbf{Z}\mathbf{E}_8 = L_0 \subset L$ , and hence a decomposition  $L = L_0 \boxplus L_1$  for some (even) unimodular  $L_1$  of rank 24. Hence, we assume throughout that  $R$  has the form

$$R = \mathbf{A}_{l_1} \boxplus \dots \boxplus \mathbf{A}_{l_r} \boxplus \mathbf{D}_{m_1} \boxplus \dots \boxplus \mathbf{D}_{m_s} \boxplus m\mathbf{E}_6 \boxplus n\mathbf{E}_7 ,$$

with no factor of type  $\mathbf{E}_8$ .

Altogether there are  $N = 88523$  such systems (of rank 32). The possible dimensions for  $m\mathbf{E}_6 \boxplus n\mathbf{E}_7$  are

$$D = \{0, 6, 7, 12, 13, 14, 18, 19, 20, 21, 24, 25, 26, 27, 28, 30, 31, 32\}$$

and for  $d \in D$ , there is a unique pair  $(m, n)$  such that  $d = 6m + 7n$ . Hence

$$N = \sum_{d \in D} \sum_{i=0}^{32-d} p(i)q(32-d-i) ,$$

where  $p(i)$  is the number of partitions of  $i$  and  $q(j)$  is the number of partitions  $(j_1, \dots, j_t)$  of  $j$  with  $4 \leq j_1 \leq \dots \leq j_t$ . (Of course, we use the convention  $p(0) = q(0) = 1$ .)

Among these, only 21209 have an acceptable deficiency, i. e.  $d = 0, 8, 12, 14, 15$  or 16. They are distributed as follows: