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Autor: Kervaire, Michel
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The associated Witt class is

$$w(\mathbf{E}_6) = \langle 1 \rangle \quad \text{in} \quad W(\mathbf{F}_3) \, .$$

CASE $R = \mathbf{E}_7$.

The definition is

$$\mathbf{Z}\mathbf{E}_7 = \{ \textstyle\sum_{i=1}^{8} x_i e_i : 2x_i \in \mathbf{Z}, \; x_i - x_j \in \mathbf{Z}, \; \sum_{i=1}^{8} x_i = 0 \} \, .$$

Here,

$$(\mathbf{Z}\mathbf{E}_7)^{\#} = \mathbf{Z}\mathbf{E}_7 \sqcup (\mathbf{Z}\mathbf{E}_7 + z_1) \, ,$$

where

$$z_1 = \tfrac{1}{4} \left(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3(e_7 + e_8) \right)$$

satisfies $(z_1, z_1) = \frac{3}{2}$ and is of minimal scalar square in its class $mod\, \mathbf{Z}\mathbf{E}_7$.

Again, z_1 is noted $x_1(\mathbf{E}_7)$ if convenient.

The Witt class $w(\mathbf{E}_7)$ is the generator $\langle 1 \rangle$ of $W(\mathbf{F}_2) = \mathbf{Z}/2\mathbf{Z}$.

CASE $R = \mathbf{E}_8$.

Here, $T(\mathbf{E}_8) = 0$. The associated Witt class is 0.

4. WEIGHT ENUMERATORS OF FINITE SCALAR PRODUCT MODULES

Let T be a finite abelian group with a non-degenerate bilinear form $b : T \times T \to \mathbf{Q}/\mathbf{Z}$.

Suppose that we have a decomposition of T as an orthogonal direct sum of subgroups $T_1, \ldots, T_s$:

$$T = T_1 \boxplus T_2 \boxplus \ldots \boxplus T_s \, .$$

Then we can define the weight $x^{w(u)} \in \mathbf{Z}[x_1, \ldots, x_s]$ of an element $u \in T$ by tabulating its non-zero components in the decomposition $u = u_1 + u_2 + \ldots + u_s$, $u_i \in T_i$, as

$$x^{w(u)} = x_1^{w(u_1)} \cdot x_2^{w(u_2)} \cdot \ldots \cdot x_s^{w(u_s)} \, ,$$

where

$$w(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{if } u_i \neq 0 \, . \end{cases}$$

If M is a subset of T, the *weight enumerator* of M is the polynomial

$$P_M(x_1, \dots, x_s) = \sum_{u \in M} x^{w(u)} .$$

We denote by q_i, $i = 1, \dots, s$ the order of the subgroup T_i.

We show in this section that MacWilliams duality is still valid in this more general setting:

THEOREM. *Let $M \subset T$ be a subgroup of the scalar product module $T = T_1 \boxplus T_2 \boxplus \dots \boxplus T_s$. Set $q_i = Card(T_i)$, and let $M^\perp$ be the subgroup orthogonal to M. Then, we have the formula, where $|M| = Card(M)$:*

$$P_{M^\perp}(x_1, \dots, x_s) =$$
$$\frac{1}{|M|} \prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_s}{1 + (q_s - 1)x_s} \right) .$$

Note that if some of the subgroups $T_1, \dots, T_s$ are mutually isomorphic (or more generally have the same order), then we can write the decomposition of T in the form

$$T = n_1 T_1 \boxplus n_2 T_2 \boxplus \dots \boxplus n_r T_r ,$$

where $n_i T_i$ stands for the orthogonal sum

$$n_i T_i = T_i \boxplus T_i \boxplus \dots \boxplus T_i$$

of n_i copies of T_i.

The weight of an element

$$u = (u_{1,1} + \dots + u_{1,n_1}) + \dots + (u_{r,1} + \dots + u_{r,n_r})$$

is then defined as

$$x^{w(u)} = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdot \dots \cdot x_r^{\nu_r} ,$$

where ν_i is the number of non-zero components of $u_{i,1} + \dots + u_{i,n_i}$ in $n_i T_i$.

The duality theorem then takes the seemingly more general form

$$P_{M^\perp}(x_1, \dots, x_r) =$$
$$\frac{1}{Card(M)} \prod_{i=1}^{r} (1 + (q_i - 1)x_i)^{n_i} \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_r}{1 + (q_r - 1)x_r} \right).$$

This identity can be viewed as a system of linear equations for the coefficients of the weight enumerator polynomial P_M of any putative metabolizer $M = M^\perp$. If M exists, this system must be solvable in non-negative integers.

Proof of the duality theorem. One of the classical proofs of MacWilliams duality in a vector space over a finite field goes over with only insignificant changes. We repeat the argument for the reader's convenience.

Let $\chi : \mathbf{Q}/\mathbf{Z} \to \mathbf{C}^*$ be the character given by $\chi(\alpha) = e^{2\pi i \alpha}$. Set $\beta(u, v) = \chi(b(u, v))$.

We cook up the function $f: T \to \mathbf{C}[x_1, \ldots, x_s]$ given by

$$f(u) = \sum_{v \in T} \beta(u, v) \cdot x^{w(v)}$$

and evaluate $\sum_{u \in M} f(u)$ in two different ways, using the following lemma:

LEMMA.

$$\sum_{u \in M} \beta(u, v) = \begin{cases} Card(M) & \text{if } v \in M^{\perp}, \\ 0 & \text{if } v \notin M^{\perp}. \end{cases}$$

We first recall the proof of the lemma.

If $v \in M^{\perp}$, then $\beta(u, v) = 1$ for every $u \in M$, thus $\sum_{u \in M} \beta(u, v) = Card(M)$ as stated in this case.

If $v \notin M^{\perp}$, there is an element $u_1 \in M$ such that $b(u_1, v) \neq 0$, and then $\beta(u_1, v) \neq 1$. We have

$$\sum_{u \in M} \beta(u, v) = \sum_{u \in M} \beta(u_1 + u, v)$$
$$= \sum_{u \in M} \beta(u_1, v) \beta(u, v) = \beta(u_1, v) \sum_{u \in M} \beta(u, v) .$$

This implies the statement of the lemma for $v \notin M^{\perp}$.

We now proceed to the proof of the duality theorem.

Firstly,

$$\sum_{u \in M} f(u) = \sum_{u \in M} \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} = \sum_{v \in T} (\sum_{u \in M} \beta(u, v)) \cdot x^{w(v)}$$
$$= \sum_{v \in M^{\perp}} Card(M) \cdot x^{w(v)} = Card(M) \cdot P_{M^{\perp}}(x_1, \ldots, x_s) .$$

Secondly,

$$f(u) = \sum_{v \in T} \beta(u, v) \cdot x^{w(v)}$$
$$= \sum_{v_1 \in T_1, \ldots, v_s \in T_s} \beta(u_1, v_1) \cdot \ldots \cdot \beta(u_s, v_s) \cdot x_1^{w(v_1)} \cdot \ldots \cdot x_s^{w(v_s)}$$
$$= \prod_{i=1}^{s} (\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)}) ,$$

where $u = u_1 + \ldots + u_s$ is the decomposition of $u \in T = T_1 \boxplus \ldots \boxplus T_s$.

Using the lemma again, we have

$$\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)} = \begin{cases} 1 + (q_i - 1)x_i & \text{if } u_i = 0, \\ 1 - x_i & \text{if } u_i \neq 0 . \end{cases}$$

Thus,

$$f(u) = \prod_{i \in S} (1 + (q_i - 1)x_i) \cdot \prod_{i \in S'} (1 - x_i) ,$$

where $S \subset \{1, ..., s\}$ is the set of indices i for which $u_i = 0$, and $S' \subset \{1, ..., s\}$ the set of indices i for which $u_i \neq 0$.

Another way of writing $f(u)$ is

$$f(u) = \prod_{i=1}^{s} (1 - x_i)^{w(u_i)} \cdot (1 + (q_i - 1)x_i)^{1 - w(u_i)} .$$

Plugging this formula into $\sum_{u \in M} f(u)$, we get

$$\begin{aligned} \textstyle\sum_{u \in M} f(u) &= \textstyle\prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot \sum_{u \in M} \prod_{i=1}^{s} \left(\frac{1 - x_i}{1 + (q_i - 1)x_i}\right)^{w(u_i)} \\ &= \textstyle\prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, ..., \frac{1 - x_s}{1 + (q_s - 1)x_s}\right) . \end{aligned}$$

Comparing the two expressions for $\sum_{u \in M} f(u)$, we get the theorem.

5. The deficiency

The main further necessary condition for a root system to be contained in an even unimodular lattice of the same rank is provided by the notion of deficiency (Defekt) introduced and studied in [KV].

If R is a root system of rank n, the *deficiency* of R, denoted $d(R)$, is the difference $n - m$, where m is the maximal cardinality of a set $\{a_1, ..., a_m\} \subset R$ of mutually orthogonal roots

$$(a_i, a_j) = 2\delta_{ij}, \quad \text{for all } 1 \leqslant i, j \leqslant m .$$

We use this notion only if all roots in R have the same scalar square 2.

If $R = R_1 \boxplus R_2$, then $d(R) = d(R_1) + d(R_2)$. The values of the deficiency for the irreducible root systems are

$$d(\mathbf{A}_l) = \left[\tfrac{l}{2}\right] ,$$

$$d(\mathbf{D}_l) = \begin{cases} 0 & \text{for } l \text{ even,} \\ 1 & \text{for } l \text{ odd ,} \end{cases}$$

$$d(\mathbf{E}_6) = 2, \ d(\mathbf{E}_7) = d(\mathbf{E}_8) = 0 .$$