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UNIMODULAR LATTICES WITH A COMPLETE ROOT SYSTEM

by Michel KERVAIRE

1. INTRODUCTION

Let $\mathbf{Q}^n$ be the n-dimensional euclidean space (over the field $\mathbf{Q}$ of rational numbers) endowed with the standard scalar product

$$(x, y) = \sum_{i=1}^{n} x_i y_i ,$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

A lattice $L \subset \mathbf{Q}^n$ is a $\mathbf{Z}$-submodule of rank n of $\mathbf{Q}^n$, i.e.

$$L = \{ \sum_{i=1}^{n} a_i v_i : a_i \in \mathbf{Z} \} ,$$

where $v_1, \dots, v_n$ is some basis of $\mathbf{Q}^n$. We are interested in *integral* lattices, i.e. lattices L satisfying $(x, y) \in \mathbf{Z}$ for all $x, y \in L$.

An integral lattice L is said to be *unimodular* if

$$det(S) = \pm 1 ,$$

where S is the $n \times n$ matrix of scalar products

$$S = ((v_i, v_j)), \quad 1 \leqslant i, j \leqslant n ,$$

$v_1, \dots, v_n$ being a $\mathbf{Z}$-basis of L. The number $det(S)$ is called the *determinant* of L and is denoted $det(L)$. It does not depend on the choice of the $\mathbf{Z}$-basis $v_1, \dots, v_n$ of L.

If L is an integral lattice, the set

$$R = \{x \in L : (x, x) = 2\}$$

is a *root system*. (For the general notion of a root system see [B], p. 142.)

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The root system R will be said to be *complete* in L if the sublattice $N = \mathbf{Z}R$ of L generated by the roots R is a subgroup of finite index in L.

Our purpose is to study unimodular lattices with a complete root system.

It is well known that there are finitely many isomorphism classes of unimodular lattices $L \subset \mathbf{Q}^n$ for a given n. (See [MH], p. 18.)

The subcollection consisting of the lattices with a complete root system is particularly interesting, e.g. in view of the connection with the theory of error-correcting codes as we shall recall below.

We begin by setting up some necessary conditions that a root system must satisfy in order to be a complete root system in a unimodular lattice (Sections 3, 4 and 5).

We are particularly interested in even unimodular lattices, i.e. (x, x) is even for every $x \in L$. In this case, as is well known, the rank of L has to be divisible by 8. In dimensions 8, 16 and 24, where the classification of even unimodular lattices is available, it turns out that every such lattice has a complete root system, with the sole exception of the 24-dimensional Leech lattice. (History and relevant literature in e.g. [N], p. 142.)

In dimension 32, there are millions of even unimodular lattices. (See [Se], p. 95.) Among them as we shall see, only a small subcollection have a complete root system. In this paper, we endeavour to provide the complete list of such lattices.

There are 132 indecomposable even unimodular 32-dimensional lattices with a complete root system. In some cases several lattices happen to have the same root system. Thus, only a total number of 119 root systems correspond to these lattices. They are listed in Section 6.

The enumeration of the lattices and their root systems could only be completed using a computer, thanks to the generous help of Shalom Eliahou who patiently explained to me the use of mulisp programming language. Of course any mistake in the programs is my sole responsibility. It is a pleasure to express to him here my warmest gratitude.

I am also deeply indebted to Boris Venkov for very valuable discussions, in particular on the use of the notion of deficiency. (See Section 5.)

2. Relationship with codes

As is customary we shall use codes to describe lattices. We briefly recall how this can be done.

If $X \subset \mathbf{Q}^n$ is any finitely generated $\mathbf{Z}$-submodule of $\mathbf{Q}^n$, we set

$$X^{\#} = \{u \in \mathbf{Q}X : (u, x) \in \mathbf{Z} \text{ for all } x \in X\} .$$

Note that if $X_1, X_2 \subset \mathbf{Q}^n$ are mutually orthogonal finitely generated $\mathbf{Z}$-submodules of $\mathbf{Q}^n$, then

$$(X_1 \boxplus X_2)^\# = X_1^\# \boxplus X_2^\# ,$$

where we write the symbol $\boxplus$ to mean orthogonal (direct) sum.

Clearly, a lattice $L \subset \mathbf{Q}^n$ is integral if and only if $L \subset L^\#$, and L is unimodular precisely if $L = L^\#$. Indeed, if $v_1, \ldots, v_n$ is a $\mathbf{Z}$-basis of L, and $w_1, \ldots, w_n$ the dual basis of $L^\#$, where $(v_i, w_j) = \delta_{i,j}$, then $v_j = \sum_{k=1}^n t_{kj} w_k$ for some integral matrix $T = (t_{kj})$, and if S is the matrix of scalar products (v_i, v_j), then

$$(v_i, v_j) = (v_i, \sum_{k=1}^{n} t_{kj} w_k) = t_{ij} ,$$

and thus

$$[L^\# : L] = |\, det(T) \,| = |\, det(S) \,| .$$

Suppose now that L is an integral lattice in $\mathbf{Q}^n$ and that $N \subset L$ is a sublattice of finite index in L. Then, $N \subset L \subset L^\# \subset N^\#$ and the finite abelian group $T(N) = N^\# / N$ inherits a non-degenerate $\mathbf{Q}/\mathbf{Z}$-valued bilinear form

$$b : T(N) \times T(N) \to \mathbf{Q}/\mathbf{Z}$$

defined by

$$b(\xi, \eta) = (x, y) \quad mod\, \mathbf{Z} ,$$

where $x, y \in N^\#$ project on $\xi, \eta \in T(N) = N^\# / N$ respectively by the natural map $\pi : N^\# \to T(N)$.

The finite scalar product module $(T(N), b)$ is called the *discriminant form* of N.

Let $M = \pi(L)$. Then M is self-orthogonal, i.e. $M \subset M^\perp$, for the bilinear form b on $T(N)$. Thus M is a self-orthogonal code in $(T(N), b)$. Conversely, given a subgroup $M \subset T(N)$ such that $M \subset M^\perp$, we recapture the integral lattice L as $L = \pi^{-1}(M)$. Note that L is unimodular, i.e. $L = L^\#$ if and only if $M = M^\perp$. If T is a finite (abelian) group with a non-degenerate bilinear form $b : T \times T \to \mathbf{Q}/\mathbf{Z}$, and $M \subset T$ is a subgroup such that $M = M^\perp$, we say that M is a *metabolizer* for the scalar product module T. A metabolizer is the same object as a self-dual code.

Summarizing, one way of describing a unimodular lattice $L \subset \mathbf{Q}^n$ consists in giving the following data:

1) An integral lattice $N \subset \mathbf{Q}^n$;

2) A metabolizer $M \subset T(N)$, where $T(N) = N^{\#}/N$ is the discriminant form of N.

We will presently make use of this: If L is a unimodular lattice with a complete root system R, then $N = \mathbf{Z}R \subset L$ is a lattice of finite index in L, and by the above, L can be encoded by the data of the root system R which determines $N = \mathbf{Z}R \subset \mathbf{Q}^n$, $N^{\#}$ and $T(R) = N^{\#}/N$ with its non-degenerate form $b : T(R) \times T(R) \to \mathbf{Q}/\mathbf{Z}$, together with a metabolizer $M = M^{\perp} \subset T(R)$.

Note however that if we start with a root system $R \subset \mathbf{Q}^n$ and construct L as $L = \pi^{-1}(M)$, where M is a metabolizer in $T(R)$, then $R' = \{a \in L : (a, a) = 2\}$ will contain R but may possibly be strictly larger.

We shall say that M is an *admissible* metabolizer if indeed we have $R = \{a \in L : (a, a) = 2\}$, where $L = \pi^{-1}(M)$.

Thus, the problem of deciding whether there exists a unimodular lattice $L \subset \mathbf{Q}^n$ with given root system R such that $\mathbf{Q}R = \mathbf{Q}^n$ is equivalent to the question: Does the finite scalar product module $(T(R), b)$ possess an admissible metabolizer?

If $R \subset \mathbf{Q}^n$ is a root system and $N = \mathbf{Z}R \subset N^{\#}$ is the lattice generated by R, we define the *norm*

$$\mathbf{n} : T(R) \to \mathbf{Q}$$

by $\mathbf{n}(\xi) = min\{(x, x) : \pi(x) = \xi\}$, where the minimum of (x, x) is taken over all the elements $x \in N^{\#}$ representing $\xi \in T(R) = N^{\#}/N$.

We say that ξ is *admissible* if $\mathbf{n}(\xi) = 0$, or $\mathbf{n}(\xi)$ is an integer > 2. It is easy to see that a metabolizer $M \subset T(R)$ is admissible if and only if every $\xi \in M$ is admissible. (Note that if R is a complete root system in L, then L cannot contain any vector u with $(u, u) = 1$.)

If an *even* unimodular lattice L is required with a prescribed root system R, then the metabolizer $M \subset T(R)$ will have to satisfy the additional condition: For all non-zero $\xi \in M$, the norm $\mathbf{n}(\xi)$ must be an even integer $\geqslant 4$. Depending on the context, we occasionally change the meaning of "admissible" to include this stronger condition, e.g. in Section 6, when setting up the tables of even unimodular lattices in dimension 32.

The classification of root systems is well known. (See [B], p. 197.) We recall the facts which are relevant to us in the next section, following mostly the notations of [N]. The possible lattices $N = \mathbf{Z}R$ are thus easily described as well as the finite scalar product modules $T(R) = N^{\#}/N$.

The existence of a mere metabolizer for $(T(R), b)$, perhaps not admissible, is already a strong restriction on R. We study this condition in the next Section 3.

We give some necessary conditions for the existence of an admissible metabolizer using coding theory in Section 4.

In Section 6, after explaining the notations used in the tables, we list the even unimodular lattices with complete root systems in dimension 32.

3. THE WITT CLASS ASSOCIATED WITH A ROOT SYSTEM

Recall the Witt group $W(\mathbf{Q}/\mathbf{Z})$ of finite scalar product modules: If T and T' are two finite abelian groups with non-degenerate bilinear forms $b: T \times T \to \mathbf{Q}/\mathbf{Z}$, $b': T' \times T' \to \mathbf{Q}/\mathbf{Z}$, then T and T' are said to be Witt equivalent if there exist finite scalar product modules H, H' each with a metabolizer $M = M^{\perp} \subset H$, $M' = M'^{\perp} \subset H'$ such that $T \boxplus H$ and $T' \boxplus H'$ are isometric. The Witt equivalence classes of finite scalar product modules form an abelian group $W(\mathbf{Q}/\mathbf{Z})$ under the operation induced by orthogonal direct sum $\boxplus$.

We recall below the explicit determination of $W(\mathbf{Q}/\mathbf{Z})$.

Let $R \subset \mathbf{Q}^n$ be a root system. As before, we denote by $T(R)$ the associated finite scalar product module. As a group, $T(R) = (\mathbf{Z}R)^{\#}/\mathbf{Z}R$, where

$$(\mathbf{Z}R)^{\#} = \{v \in \mathbf{Q}R = \mathbf{Q}^n : (v, R) \subset \mathbf{Z}\} .$$

The bilinear form $b: T(R) \times T(R) \to \mathbf{Q}/\mathbf{Z}$ is induced from the scalar product in $\mathbf{Q}^n$, restricted to $(\mathbf{Z}R)^{\#}$.

The Witt class of $(T(R), b)$ is an element of $W(\mathbf{Q}/\mathbf{Z})$ which we call the Witt class associated with the root system R and denote by $w(R) \in W(\mathbf{Q}/\mathbf{Z})$.

As we saw in Section 2, if R is the root system of a unimodular lattice $L \subset \mathbf{Q}^n$, and R is complete in L, i.e. $\mathbf{Q}R = \mathbf{Q}L = \mathbf{Q}^n$, then $(T(R), b)$ possesses a metabolizer and therefore $w(R)$ must be 0 in $W(\mathbf{Q}/\mathbf{Z})$.

If $R = R_1 \boxplus R_2$ is an orthogonal decomposition of the root system R, i.e. if R is the disjoint union $R_1 \sqcup R_2$ of two mutually orthogonal root systems R_1, R_2, then

$$w(R) = w(R_1) + w(R_2) .$$

Indeed,

$$(\mathbf{Z}R)^{\#} = (\mathbf{Z}R_1)^{\#} \boxplus (\mathbf{Z}R_2)^{\#} ,$$

and $T(R)$ is the direct product of the two subgroups $T(R_1)$ and $T(R_2)$ which are mutually orthogonal under the form b.

Now, any root system is an orthogonal sum of uniquely determined indecomposable root systems. It is therefore sufficient to calculate the Witt class associated with the indecomposable orthogonal summands.

As is well known, the list of indecomposable root systems (in which every root has scalar square 2) consists of the two infinite families $\mathbf{A}_l$, $l \geqslant 1$ and $\mathbf{D}_l$, $l \geqslant 4$ and of three exceptional systems $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$. In each case the index indicates the rank, i.e. $dim_{\mathbf{Q}}\mathbf{Q}R$. (See [B].)

If the decomposition of the root system R contains a_i copies of the indecomposable system R_i, $i = 1, \ldots, r$, we write

$$R = a_1 R_1 \boxplus a_2 R_2 \boxplus \ldots \boxplus a_r R_r \, .$$

By the above, we have

$$w(R) = \textstyle\sum_{i=1}^{r} a_i w(R_i) \in W(\mathbf{Q}/\mathbf{Z}) \, ,$$

and $w(R) = 0$ *is a necessary condition for* R *to be the complete root system of a unimodular lattice.*

In order to evaluate $w(R)$ for a given root system R, we have to determine the Witt classes $w(\mathbf{A}_l)$, $w(\mathbf{D}_l)$ and $w(\mathbf{E}_l)$ in $W(\mathbf{Q}/\mathbf{Z})$ associated with the indecomposable root systems. This is the purpose of this section.

We first briefly recall the calculation of $W(\mathbf{Q}/\mathbf{Z})$. (See [Sch], p. 166-170 for more details.)

THEOREM. $W(\mathbf{Q}/\mathbf{Z}) = \oplus_{p \in P} W(\mathbf{F}_p)$, *where* $P = \{2, 3, 5, \ldots\}$ *is the set of prime numbers, and where* $W(\mathbf{F}_p)$ *is the Witt group of the finite field* $\mathbf{F}_p$.

$$W(\mathbf{F}_2) = \mathbf{Z}/2\mathbf{Z} \, ,$$

where the generator, denoted $\langle 1 \rangle$, *is represented by the finite group* $T = \mathbf{Z}/2\mathbf{Z}$ *endowed with the bilinear form* $b : T \times T \to \mathbf{Q}/\mathbf{Z}$ *determined by* $b(1, 1) = \frac{1}{2} \bmod \mathbf{Z}$.

For p *an odd prime, we have*

$$W(\mathbf{F}_p) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \quad \textit{if} \quad p \equiv 1 \mod 4 \, .$$

The group $W(\mathbf{F}_p)$ *is generated in this case by the classes, denoted* $\langle 1 \rangle$ *and* $\langle \varepsilon \rangle$, *of* (T, b), (T', b'), *where as finite groups* $T = T' = \mathbf{F}_p$ *and* b, b' *are respectively determined by*

$$b(1, 1) = \tfrac{1}{p} \bmod \mathbf{Z} \, , \qquad b'(1, 1) = \tfrac{\varepsilon}{p} \bmod \mathbf{Z} \, ,$$

where $\varepsilon \in \mathbf{Z}$ *is a non-square* $\bmod p\mathbf{Z}$. *(The class of* b' *is of course independent of the choice of* ε.*)*

$$W(\mathbf{F}_p) = \mathbf{Z}/4\mathbf{Z} \quad \text{if } p \equiv -1 \mod 4 .$$

The group $W(\mathbf{F}_p)$ *is generated in this case by the class, denoted* $\langle 1 \rangle$, *of* (T, b), *where* $T = \mathbf{F}_p$ *and* b *is the bilinear form determined by*

$$b(1, 1) = \tfrac{1}{p} \mod \mathbf{Z} .$$

Proof. For every finite scalar product module (T, b), we have an obvious orthogonal sum decomposition

$$(T, b) = \boxplus_{p \in P(T)} (T_p, b_p) ,$$

where $P(T)$ is the set of primes dividing the order of T and T_p is the p-primary subgroup of T (consisting of the elements whose order is a power of p), and where b_p is the restriction of b to the subgroup T_p.

It follows that

$$W(\mathbf{Q}/\mathbf{Z}) = \oplus_{p \in P} W_p ,$$

where W_p is the Witt group of finite scalar product modules (T, b), where T is a p-group and $b : T \times T \to \mathbf{Z}\left[\frac{1}{p}\right]/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$ is a non-degenerate bilinear form.

The isomorphism $W_p = W(\mathbf{F}_p)$, where $W(\mathbf{F}_p)$ is the Witt group of the finite field $\mathbf{F}_p$ is a consequence of the following lemma: If (T, b) is a finite scalar product module and $U \subset T$ is a subgroup of T, let $U^\perp$ denote the orthogonal subgroup of U, i.e. $U^\perp = \{x \in T : b(x, U) = 0\}$.

LEMMA. *With these notations, suppose that* $U \subset T$ *is a self-orthogonal subgroup of* T, *i.e.* $U \subset U^\perp$. *Let* $T' = U^\perp / U$. *Then the form* b *induces on* T' *a non-degenerate bilinear form* $b' : T' \times T' \to \mathbf{Q}/\mathbf{Z}$ *and* (T, b), (T', b') *represent the same Witt class.*

Proof. Consider the scalar product module

$$(T, b) \boxplus (T', -b') = (T \oplus T', b \oplus (-b')) .$$

The subgroup $M = f(U^\perp)$, where $f : U^\perp \to T \oplus T'$ is given by $f(x) = (x, x')$, with x' the class of $x \in U^\perp$ modulo U, is a metabolizer.

It follows that $(T, b) \boxplus (T', -b') \sim \mathbf{O}$, where $\sim$ denotes Witt equivalence and $\mathbf{O}$ on the right hand side is the trivial scalar product module.

Hence,

$$(T, b) \boxplus (T', -b') \boxplus (T', b') \sim (T', b') .$$

Since $(T', -b') \boxplus (T', b') \sim \mathbf{O}$, the lemma follows. $\square$

It is easy to see by induction on the order of T that this lemma implies $W_p = W(\mathbf{F}_p)$.

Finally, the asserted values of $W(\mathbf{F}_p)$ for the various primes p result from the classification of inner product spaces over finite fields. See for instance [MH, p. 87, Lemma 1.5]. $\square$

In concrete examples, such as the scalar product module $(T(R), b)$ associated with a root system R, the above lemma enables us to find the Witt class $w(R) \in W(\mathbf{Q}/\mathbf{Z})$ by explicit calculation.

CASE $R = \mathbf{A}_l$.

Here,

$$\mathbf{ZA}_l = \{ \textstyle\sum_{i=0}^{l} x_i e_i : x_i \in \mathbf{Z},\ \sum_{i=0}^{l} x_i = 0\} \subset \mathbf{Q}^{l+1} \ ,$$

where $e_0, e_1, \ldots, e_l$ is the standard basis of $\mathbf{Q}^{l+1}$, such that $(e_i, e_j) = \delta_{ij}$.

The root system proper $\mathbf{A}_l$ is the set $\{e_i - e_j : i \neq j\}$ of vectors in $\mathbf{ZA}_l$ with square length 2.

It is well known and easy to verify that the coset decomposition of $(\mathbf{ZA}_l)^{\#}$ modulo $\mathbf{ZA}_l$ reads

$$(\mathbf{ZA}_l)^{\#} = \bigsqcup_{r=0}^{l} (\mathbf{ZA}_l + x_r) \ ,$$

where

$$x_r = \tfrac{r}{l+1} \textstyle\sum_{i=0}^{l-r} e_i - \tfrac{l-r+1}{l+1} \sum_{j=l-r+1}^{l} e_j \ .$$

Whenever the root system $\mathbf{A}_l$ has to be specified in the notation, we denote x_r by $x_r(\mathbf{A}_l)$.

The group $T(\mathbf{A}_l) = (\mathbf{ZA}_l)^{\#}/\mathbf{ZA}_l$ is cyclic of order $l+1$, generated by the class of x_1 modulo $\mathbf{ZA}_l$.

An easy calculation shows that

$$(x_r, x_r) = \tfrac{r(l-r+1)}{l+1} \ ,$$

and in fact, this number is the minimum of the scalar square of any vector in the class of x_r modulo $\mathbf{ZA}_l$. Thus $\mathbf{n}(x_r) = \frac{r(l-r+1)}{l+1}$ for $r = 0, 1, \ldots, l$, where $\mathbf{n}(x_r)$ is the norm of x_r, as defined in Section 2.

Let p be a prime and let e be the exponent of the largest power of p dividing $l+1$. Set $q = p^e$ and $s = (l+1)/q$, prime to p.

The p-primary subgroup T_p of $T(\mathbf{A}_l)$ is cyclic of order q generated by the class of x_s modulo $\mathbf{ZA}_l$. The scalar square of this element is

$$(x_s, x_s) = \tfrac{s(l-s+1)}{l+1} = -\tfrac{s}{q} \ mod\, \mathbf{Z} \ .$$

Thus we have to calculate the Witt class represented by a cyclic p-group with non-degenerate bilinear form.

Let T be the cyclic group $\mathbf{Z}/q\mathbf{Z}$, where $q = p^e$ is a power of the prime p. Let a be an integer prime to p and let

$$b : T \times T \to \mathbf{Z}\left[\tfrac{1}{p}\right]/\mathbf{Z}$$

be the bilinear form on T determined by

$$b(1, 1) = \tfrac{a}{q} \; mod\, \mathbf{Z} .$$

Then the Witt class of (T, b) in $W(\mathbf{F}_p)$ is given by

$$w(T, b) = \begin{cases} \langle a \rangle & \text{if } e \text{ is odd,} \\ 0 & \text{if } e \text{ is even,} \end{cases}$$

where $\langle a \rangle$ is the Witt class in $W(\mathbf{F}_p)$ of the form b on $\mathbf{F}_p$ given by $b(1, 1) = \frac{a}{p}\; mod\, \mathbf{Z}$.

Indeed, if e is even, $e = 2f$, then the subgroup generated by p^f in $\mathbf{Z}/q\mathbf{Z}$ is a metabolizer. If $e = 2f - 1$, let $U = p^f\mathbf{Z}/q\mathbf{Z}$ be the subgroup generated by p^f. Then, $U^\perp = p^{e-f}\mathbf{Z}/q\mathbf{Z} = p^{f-1}\mathbf{Z}/q\mathbf{Z}$. The quotient $T' = U^\perp/U$ with the induced form is isomorphic, as a scalar product module, to $\mathbf{F}_p$ with the form given by $(1, 1) = \frac{a}{p}$. By the lemma above, (T, b) and (T', b') belong to the same Witt class. The result follows.

Applying this to our example arising from the root system $\mathbf{A}_l$ with $T(\mathbf{A}_l) = \mathbf{Z}/(l + 1)\mathbf{Z}$, $q = p^e$ the exact power of p dividing $l + 1$ and $s = (l + 1)/q$, we get:

The p-component of the Witt class associated with $\mathbf{A}_l$ is

$$w_p(\mathbf{A}_l) = \begin{cases} \langle -s \rangle & \text{if } e = v_p(l + 1) \text{ is odd,} \\ 0 & \text{if } e = v_p(l + 1) \text{ is even,} \end{cases}$$

where $e = v_p(l + 1)$ is the exponent of the exact power of p dividing $l + 1$.

Note that for $p \equiv 1 \; mod\; 4$,

$$\langle -s \rangle = \langle s \rangle = \langle 1 \rangle, \text{ resp. } \langle \varepsilon \rangle$$

in $W(\mathbf{F}_p) = \mathbf{Z}/2\mathbf{Z}\langle 1 \rangle \oplus \mathbf{Z}/2\mathbf{Z}\langle \varepsilon \rangle$ depending on whether s is or is not a square $mod\, p$ respectively.

For $p \equiv -1 \; mod\; 4$, then

$$\langle -s \rangle = \langle 1 \rangle \text{ in } W(\mathbf{F}_p) = \mathbf{Z}/4\mathbf{Z}\langle 1 \rangle ,$$

if $-s$ is a square $mod\, p$, and

$$\langle -s \rangle = \langle -1 \rangle = -\langle 1 \rangle \text{ in } W(\mathbf{F}_p) = \mathbf{Z}/4\mathbf{Z}\langle 1 \rangle ,$$

if $-s$ is a non-square $mod\, p$.

CASE $R = \mathbf{D}_l$.

By definition

$$\mathbf{ZD}_l = \{ \sum_{i=1}^{l} x_i e_i : x_i \in \mathbf{Z},\ \sum_{i=1}^{l} x_i \equiv 0 \ mod\, 2\mathbf{Z}\} .$$

It is easy to check that

$$(\mathbf{ZD}_l)^{\#} = \{ \sum_{i=1}^{l} \xi_i e_i : \xi_i \in \tfrac{1}{2}\mathbf{Z},\ \xi_1 \equiv \xi_2 \equiv \dots \equiv \xi_l \, mod\, \mathbf{Z}\} ,$$

and thus

$$T(\mathbf{D}_l) = (\mathbf{ZD}_l)^{\#}/\mathbf{ZD}_l = \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } l \text{ is even,} \\ \mathbf{Z}/4\mathbf{Z} & \text{if } l \text{ is odd.} \end{cases}$$

In this case, the associated finite scalar product module $T(\mathbf{D}_l)$ always represents 0 in the Witt group $W(\mathbf{Q}/\mathbf{Z})$.

The coset decomposition of $(\mathbf{ZD}_l)^{\#}$ modulo $\mathbf{ZD}_l$ is

$$(\mathbf{ZD}_l)^{\#} = \mathbf{ZD}_l \sqcup (\mathbf{ZD}_l + y_1) \sqcup (\mathbf{ZD}_l + y_2) \sqcup (\mathbf{ZD}_l + y_3) ,$$

with

$$y_1 = \tfrac{1}{2} \sum_{i=1}^{l} e_i ,$$
$$y_2 = e_l ,$$
$$y_3 = \tfrac{1}{2} (\sum_{i=1}^{l-1} e_i - e_l) ,$$

and y_1, y_2, y_3 as above are of minimal square length in their class $mod\, \mathbf{ZD}_l$. Therefore, $\mathbf{n}(y_1) = \mathbf{n}(y_3) = \frac{l}{4}$ and $\mathbf{n}(y_2) = 1$.

When we need to include the root system in the notations, we write $x_k(\mathbf{D}_l)$ for y_k.

CASE $R = \mathbf{E}_6$.

Recall that

$$\mathbf{ZE}_6 = \{ \sum_{i=1}^{8} x_i e_i : 2x_i \in \mathbf{Z},\ x_i - x_j \in \mathbf{Z},\ \sum_{i=1}^{6} x_i = x_7 + x_8 = 0\} .$$
$$(\mathbf{ZE}_6)^{\#} = \mathbf{ZE}_6 \sqcup (\mathbf{ZE}_6 + z_1) \sqcup (\mathbf{ZE}_6 - z_1) ,$$

where

$$z_1 = \tfrac{1}{3} (e_1 + e_2 + e_3 + e_4 - 2(e_5 + e_6))$$

and $(z_1, z_1) = \frac{4}{3}$. Here again, z_1 has minimal square length in its class modulo $\mathbf{ZE}_6$ and hence $\mathbf{n}(z_1) = (z_1, z_1) = \frac{4}{3}$.

We write $x_1(\mathbf{E}_6)$ for z_1 when convenient.

The associated Witt class is

$$w(\mathbf{E}_6) = \langle 1 \rangle \quad \text{in} \quad W(\mathbf{F}_3)\,.$$

CASE $R = \mathbf{E}_7$.

The definition is

$$\mathbf{ZE}_7 = \{ \textstyle\sum_{i=1}^{8} x_i e_i : 2x_i \in \mathbf{Z},\ x_i - x_j \in \mathbf{Z},\ \sum_{i=1}^{8} x_i = 0 \}\,.$$

Here,

$$(\mathbf{ZE}_7)^{\#} = \mathbf{ZE}_7 \sqcup (\mathbf{ZE}_7 + z_1)\,,$$

where

$$z_1 = \tfrac{1}{4}\,(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3(e_7 + e_8))$$

satisfies $(z_1, z_1) = \frac{3}{2}$ and is of minimal scalar square in its class $mod\,\mathbf{ZE}_7$.

Again, z_1 is noted $x_1(\mathbf{E}_7)$ if convenient.

The Witt class $w(\mathbf{E}_7)$ is the generator $\langle 1 \rangle$ of $W(\mathbf{F}_2) = \mathbf{Z}/2\mathbf{Z}$.

CASE $R = \mathbf{E}_8$.

Here, $T(\mathbf{E}_8) = 0$. The associated Witt class is 0.

4. WEIGHT ENUMERATORS OF FINITE SCALAR PRODUCT MODULES

Let T be a finite abelian group with a non-degenerate bilinear form $b : T \times T \to \mathbf{Q}/\mathbf{Z}$.

Suppose that we have a decomposition of T as an orthogonal direct sum of subgroups $T_1, \ldots, T_s$:

$$T = T_1 \boxplus T_2 \boxplus \ldots \boxplus T_s\,.$$

Then we can define the weight $x^{w(u)} \in \mathbf{Z}[x_1, \ldots, x_s]$ of an element $u \in T$ by tabulating its non-zero components in the decomposition $u = u_1 + u_2 + \ldots + u_s$, $u_i \in T_i$, as

$$x^{w(u)} = x_1^{w(u_1)} \cdot x_2^{w(u_2)} \cdot \ldots \cdot x_s^{w(u_s)}\,,$$

where

$$w(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{if } u_i \neq 0\,. \end{cases}$$

If M is a subset of T, the *weight enumerator* of M is the polynomial

$$P_M(x_1, \ldots, x_s) = \sum_{u \in M} x^{w(u)} .$$

We denote by q_i, $i = 1, \ldots, s$ the order of the subgroup T_i.

We show in this section that MacWilliams duality is still valid in this more general setting:

THEOREM. *Let* $M \subset T$ *be a subgroup of the scalar product module* $T = T_1 \boxplus T_2 \boxplus \ldots \boxplus T_s$. *Set* $q_i = Card(T_i)$, *and let* $M^\perp$ *be the subgroup orthogonal to* M. *Then, we have the formula, where* $|M| = Card(M)$:

$$P_{M^\perp}(x_1, \ldots, x_s) =$$
$$\frac{1}{|M|} \prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \ldots, \frac{1 - x_s}{1 + (q_s - 1)x_s} \right) .$$

Note that if some of the subgroups $T_1, \ldots, T_s$ are mutually isomorphic (or more generally have the same order), then we can write the decomposition of T in the form

$$T = n_1 T_1 \boxplus n_2 T_2 \boxplus \ldots \boxplus n_r T_r ,$$

where $n_i T_i$ stands for the orthogonal sum

$$n_i T_i = T_i \boxplus T_i \boxplus \ldots \boxplus T_i$$

of n_i copies of T_i.

The weight of an element

$$u = (u_{1,1} + \ldots + u_{1,n_1}) + \ldots + (u_{r,1} + \ldots + u_{r,n_r})$$

is then defined as

$$x^{w(u)} = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdot \ldots \cdot x_r^{\nu_r} ,$$

where ν_i is the number of non-zero components of $u_{i,1} + \ldots + u_{i,n_i}$ in $n_i T_i$.

The duality theorem then takes the seemingly more general form

$$P_{M^\perp}(x_1, \ldots, x_r) =$$
$$\frac{1}{Card(M)} \prod_{i=1}^{r} (1 + (q_i - 1)x_i)^{n_i} \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \ldots, \frac{1 - x_r}{1 + (q_r - 1)x_r} \right).$$

This identity can be viewed as a system of linear equations for the coefficients of the weight enumerator polynomial P_M of any putative metabolizer $M = M^\perp$. If M exists, this system must be solvable in non-negative integers.

Proof of the duality theorem. One of the classical proofs of MacWilliams duality in a vector space over a finite field goes over with only insignificant changes. We repeat the argument for the reader's convenience.

Let $\chi : \mathbf{Q}/\mathbf{Z} \to \mathbf{C}^*$ be the character given by $\chi(\alpha) = e^{2\pi i \alpha}$. Set $\beta(u, v) = \chi(b(u, v))$.

We cook up the function $f : T \to \mathbf{C}[x_1, \ldots, x_s]$ given by

$$f(u) = \sum_{v \in T} \beta(u, v) \cdot x^{w(v)}$$

and evaluate $\sum_{u \in M} f(u)$ in two different ways, using the following lemma:

LEMMA.

$$\sum_{u \in M} \beta(u, v) = \begin{cases} Card(M) & \text{if } v \in M^{\perp}, \\ 0 & \text{if } v \notin M^{\perp}. \end{cases}$$

We first recall the proof of the lemma.

If $v \in M^{\perp}$, then $\beta(u, v) = 1$ for every $u \in M$, thus $\sum_{u \in M} \beta(u, v) = Card(M)$ as stated in this case.

If $v \notin M^{\perp}$, there is an element $u_1 \in M$ such that $b(u_1, v) \neq 0$, and then $\beta(u_1, v) \neq 1$. We have

$$\begin{aligned} \sum_{u \in M} \beta(u, v) &= \sum_{u \in M} \beta(u_1 + u, v) \\ &= \sum_{u \in M} \beta(u_1, v) \beta(u, v) = \beta(u_1, v) \sum_{u \in M} \beta(u, v) . \end{aligned}$$

This implies the statement of the lemma for $v \notin M^{\perp}$.

We now proceed to the proof of the duality theorem.

Firstly,

$$\begin{aligned} \sum_{u \in M} f(u) &= \sum_{u \in M} \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} = \sum_{v \in T} \left(\sum_{u \in M} \beta(u, v)\right) \cdot x^{w(v)} \\ &= \sum_{v \in M^{\perp}} Card(M) \cdot x^{w(v)} = Card(M) \cdot P_{M^{\perp}}(x_1, \ldots, x_s) . \end{aligned}$$

Secondly,

$$\begin{aligned} f(u) &= \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} \\ &= \sum_{v_1 \in T_1, \ldots, v_s \in T_s} \beta(u_1, v_1) \cdot \ldots \cdot \beta(u_s, v_s) \cdot x_1^{w(v_1)} \cdot \ldots \cdot x_s^{w(v_s)} \\ &= \prod_{i=1}^{s} \left(\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)}\right) , \end{aligned}$$

where $u = u_1 + \ldots + u_s$ is the decomposition of $u \in T = T_1 \boxplus \ldots \boxplus T_s$.

Using the lemma again, we have

$$\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)} = \begin{cases} 1 + (q_i - 1) x_i & \text{if } u_i = 0, \\ 1 - x_i & \text{if } u_i \neq 0 . \end{cases}$$

Thus,

$$f(u) = \prod_{i \in S} (1 + (q_i - 1)x_i) \cdot \prod_{i \in S'} (1 - x_i) ,$$

where $S \subset \{1, ..., s\}$ is the set of indices i for which $u_i = 0$, and $S' \subset \{1, ..., s\}$ the set of indices i for which $u_i \neq 0$.

Another way of writing $f(u)$ is

$$f(u) = \prod_{i=1}^{s} (1 - x_i)^{w(u_i)} \cdot (1 + (q_i - 1)x_i)^{1 - w(u_i)} .$$

Plugging this formula into $\sum_{u \in M} f(u)$, we get

$$\begin{aligned} \textstyle\sum_{u \in M} f(u) &= \textstyle\prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot \sum_{u \in M} \prod_{i=1}^{s} \left(\frac{1 - x_i}{1 + (q_i - 1)x_i}\right)^{w(u_i)} \\ &= \textstyle\prod_{i=1}^{s} (1 + (q_i - 1)x_i) \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, ..., \frac{1 - x_s}{1 + (q_s - 1)x_s}\right) . \end{aligned}$$

Comparing the two expressions for $\sum_{u \in M} f(u)$, we get the theorem.

5. The deficiency

The main further necessary condition for a root system to be contained in an even unimodular lattice of the same rank is provided by the notion of deficiency (Defekt) introduced and studied in [KV].

If R is a root system of rank n, the *deficiency* of R, denoted $d(R)$, is the difference $n - m$, where m is the maximal cardinality of a set $\{a_1, ..., a_m\} \subset R$ of mutually orthogonal roots

$$(a_i, a_j) = 2\delta_{ij}, \qquad \text{for all } 1 \leqslant i, j \leqslant m .$$

We use this notion only if all roots in R have the same scalar square 2.

If $R = R_1 \boxplus R_2$, then $d(R) = d(R_1) + d(R_2)$. The values of the deficiency for the irreducible root systems are

$$d(\mathbf{A}_l) = \left[\tfrac{l}{2}\right] ,$$

$$d(\mathbf{D}_l) = \begin{cases} 0 & \text{for } l \text{ even,} \\ 1 & \text{for } l \text{ odd ,} \end{cases}$$

$$d(\mathbf{E}_6) = 2, \ d(\mathbf{E}_7) = d(\mathbf{E}_8) = 0 .$$

By Satz 5 of [KV], if R is the (complete) root system of an even unimodular lattice of rank 32, then

$$d(R) = 0, 8, 12, 14, 15 \text{ or } 16 .$$

The proof consists in constructing from the given lattice a new lattice L, still of rank 32 and containing the orthogonal sum of $m = 32 - d(R)$ copies of $\mathbf{Z}$. Thus, $L = \mathbf{Z}^m \boxplus L_0$, where L_0 is again unimodular and of rank $d(R)$. (Hence, $\operatorname{rank}(L_0) \leqslant 16$.)

By Martin Kneser's classification of unimodular (positive definite) lattices of rank $\leqslant 16$, the rank of L_0, i.e. $d(R)$ can only take the above values. (See [Kn], Satz 1.)

In setting up the tables we conveniently use the deficiency to discriminate the various root systems R according to the value of $d(R)$.

6. THE TABLES

We now proceed to list the *indecomposable* even unimodular lattices L of rank 32 with a complete root system R.

The presence in R of a factor of type $\mathbf{E}_8$ would produce a unimodular sublattice $\mathbf{Z}\mathbf{E}_8 = L_0 \subset L$, and hence a decomposition $L = L_0 \boxplus L_1$ for some (even) unimodular L_1 of rank 24. Hence, we assume throughout that R has the form

$$R = \mathbf{A}_{l_1} \boxplus \dots \boxplus \mathbf{A}_{l_r} \boxplus \mathbf{D}_{m_1} \boxplus \dots \boxplus \mathbf{D}_{m_s} \boxplus \mathbf{mE}_6 \boxplus \mathbf{nE}_7 ,$$

with no factor of type $\mathbf{E}_8$.

Altogether there are $N = 88523$ such systems (of rank 32). The possible dimensions for $\mathbf{mE}_6 \boxplus \mathbf{nE}_7$ are

$$D = \{0, 6, 7, 12, 13, 14, 18, 19, 20, 21, 24, 25, 26, 27, 28, 30, 31, 32\}$$

and for $d \in D$, there is a unique pair (m, n) such that $d = 6m + 7n$. Hence

$$N = \textstyle\sum_{d \in D} \sum_{i=0}^{32-d} p(i) q(32 - d - i) ,$$

where $p(i)$ is the number of partitions of i and $q(j)$ is the number of partitions $(j_1, \dots, j_t)$ of j with $4 \leqslant j_1 \leqslant \dots \leqslant j_t$. (Of course, we use the convention $p(0) = q(0) = 1$.)

Among these, only 21209 have an acceptable deficiency, i.e. $d = 0, 8, 12, 14, 15$ or 16. They are distributed as follows:

Deficiency	0	8	12	14	15	16	Total
Number	347	9799	6282	3027	1523	231	21209
Number with zero Witt class	347	848	306	90	57	28	1676
Number of connected root systems with zero Witt class	347	410	108	34	24	11	934

We say that a root system R is *not connected* if $R = R_1 \sqcup R_2$ is a disjoint union of mutually orthogonal root systems R_1, R_2 such that $T(R_1)$ and $T(R_2)$ have relatively prime orders.

If $R = R_1 \sqcup R_2$ is not connected, a metabolizer for $T(R) = T(R_1) \boxplus T(R_2)$ will have the form $M = M_1 \boxplus M_2$, where M_i is a metabolizer for $T(R_i)$, $i = 1, 2$ and any lattice L with (complete) root system R will split as $L = L_1 \boxplus L_2$, with L_1, L_2 unimodular and with root systems R_1, R_2 respectively. Thus, if R is not connected, it does not qualify as a candidate root system for an indecomposable unimodular lattice of the same rank.

Sifting the root systems for the purpose of setting up the tables, we retain only the connected ones. Of course, a decomposable 32-dimensional lattice which does not involve a $\mathbf{ZE}_8$ factor can only be the orthogonal sum of 2 copies of the indecomposable 16-dimensional lattice Γ_{16} in the notation of [MH], Lemma 6.1, p. 27. However, the criterion is a handy one to include in a computer program and it does considerably shorten the lists of candidates. The number of remaining systems is shown as the last line in the above table.

In order to get some experimental estimate on the relative strengths of the various conditions we are using, let me display the (otherwise irrelevant) list of connected systems of admissible deficiencies. *(See the table next page.)*

Comparing the last lines of the two tables we see that the condition on the Witt class is fairly stronger than merely requiring the order of $T(R)$ to be an integral square. (Of course, if $T(R)$ contains a metabolizer $M = M^{\perp}$, then $|T(R)| = |M|^2$.) A simple example of a root system R with non-zero Witt class but $|T(R)|$ a square is $R = 2\mathbf{A}_5 \boxplus \mathbf{A}_8 \boxplus \mathbf{D}_4 \boxplus \mathbf{D}_8$ which is connected (and has deficiency 8). There are $1302 - 934 = 368$ such.

Deficiency	0	8	12	14	15	16	Total
Connected root systems	347	2154	1051	425	150	25	4152
Connected root systems with $\mid T(R) \mid$ *a square*	347	610	214	79	38	14	1302

The 934 root systems of the bottom row of the first table all possess a metabolizer. However, a metabolizer $M \subset T(R)$ will produce a unimodular lattice L with root system exactly R only if for each non-zero $s \in M$ the norm $\mathbf{n}(s)$ is an integer larger than 2: $\mathbf{n}(s) > 2$. (The norm has been defined in Section 2.) Moreover if L is to be an even lattice, $\mathbf{n}(s)$ must in addition be an even integer. A metabolizer M satisfying $\mathbf{n}(s) \equiv 0$ $(mod\, 2)$ and $\mathbf{n}(s) > 2$ for every $s \in M$, $s \neq 0$ will be called *admissible.*

The norms of the elements of $T(\mathbf{A}_l)$, $T(\mathbf{D}_l)$, $T(\mathbf{E}_6)$, and $T(\mathbf{E}_7)$ have been recalled in Section 3:

$$\mathbf{n}(x_r) = \tfrac{r(l+1-r)}{l+1} \text{ for } x_r \in T(\mathbf{A}_l),\ r = 0, 1, \ldots, l\,,$$

$$\mathbf{n}(y_1) = \mathbf{n}(y_3) = \tfrac{l}{4},\quad \mathbf{n}(y_2) = 1 \text{ for } T(\mathbf{D}_l)\,,$$

$$\mathbf{n}(z) = \begin{cases} \tfrac{4}{3} & \text{for } z \in T(\mathbf{E}_6),\ z \neq 0\,, \\ \tfrac{3}{2} & \text{for } z \in T(\mathbf{E}_7),\ z \neq 0\,. \end{cases}$$

Thus, the norm of any element in the discriminant $T(R)$ of a root system R can easily be calculated. Of course, in general $\mathbf{n}(s + s') \neq \mathbf{n}(s) + \mathbf{n}(s')$ for $s, s' \in T(R)$. However, $\mathbf{n}(s + s') = \mathbf{n}(s) + \mathbf{n}(s')$ holds true if s, s' belong to the discriminants $T(R_1)$, $T(R_2)$ of mutually orthogonal root sub-systems.

Only the weights of admissible elements may occur with non-vanishing coefficient in the weight enumerator polynomial P_M of a putative (admissible) metabolizer M.

Before embarking on using the duality theorem, it is possible, in some favorable cases, to eliminate a root system directly by inspection:

If $M \subset T(R)$ is an admissible metabolizer, then for every prime number p, the p-component M_p of M is an admissible metabolizer for the induced bilinear form on the p-component $T(R)_p$ of $T(R)$. There are cases of root systems R and suitable choice of p for which it is apparent that no metabolizer of $T(R)_p$ is admissible. As an example, suppose that $R = \mathbf{A}_2 \boxplus \mathbf{A}_5 \boxplus R'$, where the order of $T(R')$ is prime to 3. Then, $T(R)_3 = T(\mathbf{A}_2 \boxplus \mathbf{A}_5)_3 = T(\mathbf{A}_2) \boxplus T(\mathbf{A}_5)_3 = \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ generated by $s_1 = (1, 0)$, $s_2 = (0, 2)$, where $(1, 0)$ stands for the projection of $x_1 \in (\mathbf{ZA}_2)^{\#}$ in $T(\mathbf{A}_2) \boxplus T(\mathbf{A}_5)_3$ and $(0, 2)$ stands for the projection of $x_2 \in (\mathbf{ZA}_5)^{\#}$ in $T(\mathbf{A}_2) \boxplus T(\mathbf{A}_5)_3$ in the notations of Section 3. Now, $\mathbf{n}(s_1) = \frac{2}{3}$ and $\mathbf{n}(s_2) = \frac{4}{3}$, and for every $s \in T(\mathbf{A}_2 \boxplus \mathbf{A}_5)_3$ one has $\mathbf{n}(s) \leqslant 2$.

This argument eliminates the root systems of the form $R = X \boxplus R'$, with $T(R')$ of order prime to 3 if X is any member of the following (small but frequently arising) black list:

$$X = \mathbf{A}_2 \boxplus \mathbf{A}_5, \quad 2\mathbf{A}_2 \boxplus 2\mathbf{A}_5, \quad 2\mathbf{A}_2 \boxplus \mathbf{A}_5 \boxplus \mathbf{E}_6 .$$

Similarly, $R = \mathbf{mA}_2 \boxplus \mathbf{nA}_5 \boxplus \mathbf{A}_8 \boxplus R'$, with $T(R')$ of order prime to 3 cannot occur for any $m, n \geqslant 0$.

Indeed, for any putative admissible metabolizer M, one should have $M_3 \subset T(\mathbf{mA}_2 \boxplus \mathbf{nA}_5)_3 \boxplus 3T(\mathbf{A}_8)$ because any $s \in M_3$ with $3s \neq 0$ would produce an element $s' = 3s = (0^m, 0^n, \pm 3) \in M_3$, $s' \neq 0$, of norm $\mathbf{n}(s') = 2$, which is inacceptable.

But then $M_3' = M_3 \cap T(\mathbf{mA}_2 \boxplus \mathbf{nA}_5)_3$ would be a metabolizer in $T(\mathbf{mA}_2 \boxplus \mathbf{nA}_5)_3$, and therefore $M_0 = M \cap T(R_0)$ a metabolizer in $T(R_0)$, where $R_0 = \mathbf{mA}_2 \boxplus \mathbf{nA}_5 \boxplus R'$. (The subgroup M_3' is obviously self-orthogonal and it has the right order.) Setting $\pi_0 : (\mathbf{ZR_0})^{\#} \to T(R_0)$, the natural projection, the inverse image $L_0 = \pi_0^{-1}(M_0)$ would be a unimodular sublattice and hence an orthogonal summand of L.

If no such simple argument is available, the root system is to be tested using the duality theorem of Section 4.

For a given root system R, the coefficients in P_M of weight monomials which are not representable by any admissible elements in $T(R)$ must be 0. The duality theorem, using $M = M^{\perp}$, is then a linear system for the remaining coefficients of P_M which must be solvable in non-negative integers. In many cases, this system is not even solvable in rational numbers or if it is, some coefficients turn out to be negative or fractional. Here, all cases occur. In most of the remaining cases where the existence of the polynomial is not prohibited by MacWilliams duality, an admissible metabolizer and hence an even unimodular lattice can actually be constructed.

Completeness of the lists thus relies on a lengthy elimination procedure, let alone the heavy use of machine testing, subject to all sorts of failure. It would certainly be desirable to supply an alternate, perhaps less computational, approach.

The above classification program really begins with the root systems of deficiency 8. For the root systems of deficiency 0, there is another, fairly different method, due to H. Koch and B. Venkov, which we recall in the next paragraph.

NOTATIONS IN THE TABLES

The notation for root systems R is self-explanatory: If e.g. $R = \mathbf{8A}_1 \boxplus \mathbf{8A}_3$, then $\mathbf{Z}R$ is the orthogonal direct sum

$$\mathbf{Z}R = \mathbf{ZA}_1 \boxplus \cdots \boxplus \mathbf{ZA}_1 \boxplus \mathbf{ZA}_3 \boxplus \cdots \boxplus \mathbf{ZA}_3$$

of 8 copies of $\mathbf{ZA}_1$ and 8 copies of $\mathbf{ZA}_3$.

In order to describe a unimodular lattice L containing $\mathbf{Z}R$ we display a *filling set* S, i.e. a set of vectors in $(\mathbf{Z}R)^{\#}$ which together with $\mathbf{Z}R$ generate L. The terminology is intended to be reminiscent of the similar notion of a glueing set occuring in the paper of J. Conway and V. Pless [CP].

Let $R = R_1 \boxplus \cdots \boxplus R_r$ be the decomposition of R in irreducible components. The vectors in the filling set S contained in

$$(\mathbf{Z}R)^{\#} = (\mathbf{Z}R_1)^{\#} \boxplus \cdots \boxplus (\mathbf{Z}R_r)^{\#}$$

are specified by their coordinates in the successive $(\mathbf{Z}R_i)^{\#}$, $i = 1, \ldots, r$.

Vectors in the filling set are taken with minimal norm in their class modulo $\mathbf{Z}R$. It is thus easy to read off the norm of an element in S from its displayed expression in coordinates. If the i-th irreducible component R_i of R is $\mathbf{A}_l, \mathbf{D}_l, \mathbf{E}_6$, or $\mathbf{E}_7$, the number k as the i-th coordinate of a vector of S stands for the element noted $x_k(R_i)$ in Section 3.

In order (hopefully) to improve readability, I have separated by a semi-colon the components of a filling vector belonging to different multiple root systems. Thus, for instance $s = (1; 2; 1, 0)$ in the filling set for the root system $\mathbf{A}_3 \boxplus \mathbf{A}_{15} \boxplus \mathbf{2E}_7$, the 16-nth root system with deficiency 8 occuring in the tables, stands for the vector $s = x_1(\mathbf{A}_3) + x_2(\mathbf{A}_{15}) + x_1(\mathbf{E}_7) + 0$ in $(\mathbf{ZA}_3)^{\#} \boxplus (\mathbf{ZA}_{15})^{\#} \boxplus (\mathbf{ZE}_7)^{\#} \boxplus (\mathbf{ZE}_7)^{\#}$. Its norm is $\frac{1 \cdot 3}{4} + \frac{2 \cdot 14}{16} + \frac{3}{2} + 0 = 4$.

After the filling set, the reader will find the weight enumerator polynomial, sometimes just called the "polynomial" of the metabolizer $M = \pi(L)$, where $\pi : (\mathbf{Z}R)^{\#} \to T(R)$. The weights refer to the indicated decomposition of the root system under discussion, i.e. one variable only for each multiple factor nR_i, where R_i is irreducible. Thus, for instance, the term $56x^4y^2$ in the polynomial for $R = \mathbf{8A}_1 \boxplus \mathbf{8A}_3$ means that the metabolizer M contains

56 vectors with 4 non-zero coordinates among the first 8 corresponding to $T(\mathbf{A}_1)^8$ and 2 non-zero coordinates among the last 8 corresponding to $T(\mathbf{A}_3)^8$. As an example, we find among these vectors the images in $T(R)$ of the vectors s_4, s_5, s_6, s_7 of the filling set.

The root systems with a fixed deficiency are listed in alphabetical order.

1. Root systems with deficiency 0

This case has been treated by H. Koch and B. Venkov. (See [KV], Satz 3.) If L is an even unimodular lattice of rank 32 with a complete root system of deficiency 0, then L contains 32 mutually orthogonal vectors of scalar square 2, i.e. $a_1, \ldots, a_{32} \in L$ such that $(a_i, a_j) = 2\delta_{ij}$.

Let $N = \mathbf{Z}a_1 \boxplus \mathbf{Z}a_2 \boxplus \ldots \boxplus \mathbf{Z}a_{32}$ and let $N^{\#} = \mathbf{Z}\alpha_1 \boxplus \ldots \boxplus \mathbf{Z}\alpha_{32}$ be the dual lattice, where $\alpha_i = \frac{1}{2} a_i$.

Since $(x, u) \in \mathbf{Z}$ for all $x \in L$, $u \in N$, we have $L \subset N^{\#}$. The quotient $N^{\#}/N$ is the 32-dimensional vector space $\mathbf{F}_2^{32}$ with the standard scalar product $(\varepsilon_i, \varepsilon_j) = \frac{1}{2}\delta_{ij}$ (induced by the scalar product on $N^{\#}$), where ε_i stands for the image of α_i under the projection $\pi : N^{\#} \to N^{\#}/N$.

The image $C_L = \pi(L)$ of the lattice L is then a self-dual code (of dimension 16) in $\mathbf{F}_2^{32}$. Because L is even, it follows that C_L is a doubly-even code (i.e. all code words have a weight divisible by 4).

Now, the doubly-even self-dual codes in $\mathbf{F}_2^{32}$ have been classified by J. Conway and V. Pless in [CP]. There are 85 of them. Crossing out from this list the decomposable ones, we arrive at a list of 75 codes, and therefore 75 irreducible even unimodular lattices, corresponding to 62 root systems. For the details, see [CP] and [KV].

It turns out that all the examples of non-isomorphic even unimodular 32-dimensional lattices with the same complete root system occur in the case of deficiency 0.

The reader who wishes to see these examples explicitly must therefore turn to [CP].

In the following subsections 2 to 6, containing the list of lattices with non-zero deficiency, each realizable root system uniquely determines the lattice to which it belongs.

2. Root systems with deficiency 8

There are 29 even unimodular lattices of rank 32 having a complete root system of deficiency 8. Each lattice is uniquely determined by its root system.

(1) $$8A_1 \boxplus 8A_3$$

A filling set for the corresponding lattice consists of the following 8 vectors

$$
\begin{aligned}
s_0 &= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0;\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1),\\
s_1 &= (1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0;\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1),\\
s_2 &= (0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0;\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0),\\
s_3 &= (0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0;\ 0,\ 1,\ 3,\ 0,\ 0,\ 1,\ 3,\ 0),\\
s_4 &= (1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0;\ 0,\ 0,\ 0,\ 0,\ 2,\ 2,\ 0,\ 0),\\
s_5 &= (0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0;\ 0,\ 0,\ 0,\ 0,\ 0,\ 2,\ 2,\ 0),\\
s_6 &= (0,\ 1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 0;\ 0,\ 0,\ 2,\ 0,\ 0,\ 2,\ 0,\ 0),\\
s_7 &= (0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1;\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 2,\ 2).
\end{aligned}
$$

The weight enumerator polynomial is

$$
\begin{aligned}
P(x, y) = 1 &+ x^8 + 56x^4y^2 + 14y^4 + 112x^2y^4 + 112x^4y^4\\
&+ 112x^6y^4 + 14x^8y^4 + 896x^4y^5 + 672x^2y^6 + 56x^4y^6\\
&+ 672x^6y^6 + 896x^4y^7 + 17y^8 + 112x^2y^8 + 224x^4y^8\\
&+ 112x^6y^8 + 17x^8y^8.
\end{aligned}
$$

The (rather delicate) discussion of this root system in presented in Section 7.

(2) $$4A_1 \boxplus 4A_5 \boxplus D_8$$

Filling set $S = \langle s_1, s_2, s_3, s_4, s_5, s_6, s_7 \rangle$, where

$s_1 = (1,\ 0,\ 0,\ 0;\ 3,\ 0,\ 0,\ 0;\ 1)$, $s_2 = (0,\ 1,\ 0,\ 0;\ 0,\ 3,\ 0,\ 0;\ 1)$,
$s_3 = (0,\ 0,\ 1,\ 0;\ 0,\ 0,\ 3,\ 0;\ 1)$, $s_4 = (0,\ 0,\ 0,\ 1;\ 0,\ 0,\ 0,\ 3;\ 1)$,
$s_5 = (1,\ 1,\ 1,\ 1;\ 0,\ 0,\ 0,\ 0;\ 3)$, $s_6 = (0,\ 0,\ 0,\ 0;\ 0,\ 2,\ 2,\ 2;\ 0)$,
$s_7 = (0,\ 0,\ 0,\ 0;\ 2,\ 0,\ 2,\ 4;\ 0)$.

Polynomial

$$
\begin{aligned}
P(x, y, z) = 1 &+ 6x^2y^2 + 8y^3 + 24x^2y^3 + 24x^2y^4 + 9x^4y^4 + x^4z + 4xyz\\
&+ 4x^3yz + 6x^2y^2z + 36xy^3z + 24x^2y^3z + 36x^3y^3z\\
&+ 8x^4y^3z + 9y^4z + 32xy^4z + 24x^2y^4z + 32x^3y^4z.
\end{aligned}
$$

(3) $$2A_1 \boxplus 2A_3 \boxplus 2A_7 \boxplus D_{10}$$

Filling set

$S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$s_1 = (1,\ 0;\ 2,\ 0;\ 0,\ 0;\ 1)$, $s_2 = (0,\ 1;\ 0,\ 2;\ 0,\ 0;\ 3)$,
$s_3 = (0,\ 0;\ 1,\ 1;\ 2,\ 0;\ 2)$, $s_4 = (1,\ 1;\ 0,\ 1;\ 1,\ 1;\ 1)$.

Polynomial

$$P(x, y, z, t) = 1 + 2y^2z + 4x^2y^2z + z^2 + 4yz^2 + 8xyz^2 + 8xy^2z^2 + 4x^2y^2z^2 + 2xyt + x^2y^2t + 2x^2zt + 4xyzt + 4y^2zt + 8xy^2zt + 4xz^2t + 8yz^2t + 10xyz^2t + 12x^2yz^2t + 12y^2z^2t + 20xy^2z^2t + 9x^2y^2z^2t.$$

(4) $\mathbf{2A_1 \boxplus 2A_9 \boxplus D_{12}}$

Filling set

$$S = <(1, 0; 5, 0; 1), (0, 1; 0, 5; 1), (1, 0; 0, 5; 2), (0, 0; 2, 4; 0)>.$$

Polynomial

$$P(x, y, z) = 1 + 4y^2 + 5x^2y^2 + x^2z + 4xyz + 5y^2z + 16xy^2z + 4x^2y^2z.$$

(5) $\mathbf{A_1 \boxplus A_3 \boxplus 2A_7 \boxplus D_7 \boxplus E_7}$

Filling set $S = <s_1, s_2, s_3>$, where

$$s_1 = (1;\ 1;\ 1,\ 3;\ 0;\ 0), \quad s_2 = (0;\ 1;\ 2,\ 4;\ 1;\ 0), \quad s_3 = (1;\ 0;\ 0,\ 4;\ 0;\ 1).$$

Polynomial

$$P(x, y, z, t, u) = 1 + z^2 + 2yz^2 + 4xyz^2 + 6yzt + 2z^2t + 4xz^2t + 4yz^2t + 8xyz^2t + 2xzu + 4yz^2u + 2xyz^2u + xytu + 4xyztu + 4z^2tu + 2xz^2tu + 8yz^2tu + 5xyz^2tu.$$

(6) $\mathbf{A_1 \boxplus A_5 \boxplus A_{11} \boxplus D_5 \boxplus D_{10}}$

Filling set $S = <(1;3;0;2;2), (0;3;0;0;1), (1;0;3;1;0), (0;2;4;0;0)>$.

Polynomial

$$P(x, y, z, t, u) = 1 + 2yz + zt + 2xzt + 2yzt + 4xyzt + yu + xzu + 2yzu + 5xyzu + xtu + xytu + 2ztu + 13yztu + 10xyztu.$$

(7) $\mathbf{A_1 \boxplus A_{17} \boxplus D_{14}}$

Filling set $S = <(1;\ 0;\ 1), (0;\ 9;\ 3), (0;\ 6;\ 0)>$.

Polynomial $P(x, y, z) = 1 + 2y + xz + 3yz + 5xyz$.

(8) $\mathbf{8A_3 \boxplus 2D_4}$

Filling set $S = < s_1, s_2, s_3, s_4, s_5, s_6 >$, where

$s_1 = (1, 1, 1, 1, 1, 1, 1, 1; 0, 0)$, $s_2 = (0, 1, 0, 0, 1, 3, 2, 1; 0, 0)$,
$s_3 = (0, 0, 1, 0, 1, 0, 1. 1; 1, 0)$, $s_4 = (0, 0, 0, 1, 0, 1, 3, 3; 2, 0)$,
$s_5 = (0, 2, 0, 0, 0, 2, 0, 0; 1, 1)$, $s_6 = (0, 2, 0, 0, 2, 0, 0, 0; 2, 2)$.

Polynomial

$$\begin{aligned} P(x, y) = 1 &+ 14x^4 + 16x^5 + 16x^7 + 17x^8 + 48x^4y + 288x^6y \\ &+ 48x^8y + 12x^2y^2 + 24x^4y^2 + 240x^5y^2 + 12x^6y^2 \\ &+ 240x^7y^2 + 48x^8y^2. \end{aligned}$$

(9) $\mathbf{8A_3 \boxplus D_8}$

Filling set $S = < s_1, s_2, s_3, s_4, s_5 >$, where

$s_1 = (1, 0, 0, 0, 2, 1, 1, 1; 0)$, $s_2 = (0, 1, 0, 2, 1, 0, 1, 1; 0)$,
$s_3 = (0, 0, 1, 1, 0, 2, 1, 1; 0)$, $s_4 = (0, 0, 0, 1, 1, 1, 3, 2; 1)$,
$s_5 = (0, 0, 0, 0, 0, 0, 2, 2; 3)$.

Polynomial

$$\begin{aligned} P(x, y) = 1 &+ 14x^4 + 48x^5 + 48x^7 + 17x^8 + 4x^2y + 24x^4y + 112x^5y \\ &+ 100x^6y + 112x^7y + 32x^8y. \end{aligned}$$

(10) $\mathbf{7A_3 \boxplus D_{11}}$

Filling set $S = < s_1, s_2, s_3, s_4 >$, where

$s_1 = (1, 0, 0, 2, 1, 1, 1; 0)$, $s_2 = (0, 1, 0, 1, 2, 3, 1; 0)$,
$s_3 = (0, 0, 1, 3, 3, 2, 1; 0)$, $s_4 = (0, 0, 0, 1, 3, 1, 2; 1)$.

Polynomial

$$P(x, y) = 1 + 7x^4 + 42x^5 + 14x^7 + 7x^3y + 70x^4y + 98x^6y + 17x^7y.$$

(11) $\mathbf{6A_3 \boxplus 2D_7}$

Filling set $S = < s_1, s_2, s_3, s_4 >$, where

$s_1 = (1, 0, 0, 0, 1, 1; 0, 1)$, $s_2 = (0, 1, 0, 0, 1, 3; 1, 0)$,
$s_3 = (0, 0, 1, 0, 1, 2; 1, 1)$, $s_4 = (0, 0, 0, 1, 2, 1; 3, 1)$.

Polynomial

$$\begin{aligned} P(x, y) = 1 &+ 3x^4 + 12x^5 + 24x^3y + 12x^4y + 48x^5y + 12x^6y \\ &+ 3x^2y^2 + 24x^3y^2 + 48x^4y^2 + 36x^5y^2 + 33x^6y^2. \end{aligned}$$

(12) $$4A_3 \boxplus 4D_5$$

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$s_1 = (1, 1, 1, 1; 2, 0, 0, 0), \qquad s_2 = (1, 1, 0, 0; 1, 1, 0, 0),$$
$$s_3 = (0, 1, 3, 0; 0, 1, 1, 0), \qquad s_4 = (0, 0, 3, 1; 0, 0, 1, 1).$$

Polynomial

$$P(x, y) = 1 + x^4 + 8x^4y + 36x^2y^2 + 24x^4y^2 + 96x^3y^3 + 8x^4y^3 + y^4 + 8xy^4 + 24x^2y^4 + 8x^3y^4 + 41x^4y^4.$$

(13) $$2A_3 \boxplus 2A_7 \boxplus 2D_6$$

Filling set

$$S = \langle (1,0;1,1;1,0), (1,1;2,0;2,0), (2,0;0,0;1,1), (0,2;0,0;3,3) \rangle.$$

Polynomial

$$P(x, y, z) = 1 + 2x^2y + y^2 + 4xy^2 + 8x^2yz + 16xy^2z + 24x^2y^2z + 2xz^2 + x^2z^2 + 2yz^2 + 4xyz^2 + 8x^2yz^2 + 4y^2z^2 + 22xy^2z^2 + 29x^2y^2z^2.$$

(14) $$A_3 \boxplus A_5 \boxplus A_{11} \boxplus D_6 \boxplus E_7$$

Filling set $S = \langle s_0, s_1, s_2, s_3 \rangle$, where

$$s_0 = (1; 3; 3; 0; 1), \qquad s_1 = (2; 3; 0; 1; 0),$$
$$s_2 = (0; 0; 6; 3; 1), \qquad s_3 = (0; 2; 4; 0; 0).$$

Polynomial

$$P(x, y, z, t, u) = 1 + xz + 2yz + 2xyz + xyt + 2xzt + 3yzt + 12xyzt + 6xyzu + xtu + ytu + ztu + 2xztu + 4yztu + 9xyztu.$$

(15) $$A_3 \boxplus A_{11} \boxplus D_{12} \boxplus E_6$$

Filling set

$$S = \langle (1; 3; 2; 0), (0; 6; 1; 0), (0; 4; 0; 1) \rangle.$$

Polynomial

$$P(x, y, z, t) = 1 + xy + xz + yz + 4xyz + 2yt + 2xyt + 2yzt + 10xyzt.$$

(16) $\mathbf{A}_3 \boxplus \mathbf{A}_{15} \boxplus 2\mathbf{E}_7$

Filling set $S = <(1;\ 2;\ 1,\ 0),\ (2;\ 0;\ 1,\ 1)>$.
Weight polynomial

$P(x, y, z) = 1 + y + 2xy + 8xyz + xz^2 + 2yz^2 + xyz^2$.

(17) $4\mathbf{A}_5 \boxplus 2\mathbf{D}_6$

Filling set $S = <S_2,\ S_3>$, where

$$S_2 = <(3,0,0,0;1,2),\ (0,3,0,0;3,2),\ (0,0,3,0;2,1),\ (0,0,0,3;2,3)>,$$
$$S_3 = <(0,2,2,2;0,0),\ (2,0,2,4;0,0)>.$$

Polynomial

$$\mathrm{P}(x, y) = 1 + 8x^3 + 2x^2y + 20x^3y + 32x^4y + 4xy^2 + 4x^2y^2 + 40x^3y^2 + 33x^4y^2.$$

(18) $4\mathbf{A}_5 \boxplus \mathbf{D}_{12}$

Filling set $S = <S_2,\ S_3>$, where

$$S_2 = <(3,\ 3,\ 3,\ 3;\ 0),\ (3,\ 3,\ 0,\ 0;\ 1),\ (0,\ 3,\ 3,\ 0;\ 2)>$$
$$S_3 = <(0,\ 2,\ 2,\ 2;\ 0),\ (2,\ 0,\ 2,\ 4;\ 0)>.$$

Polynomial

$P(x, y) = 1 + 8x^3 + 9x^4 + 6x^2y + 24x^3y + 24x^4y$.

(19) $3\mathbf{A}_5 \boxplus \mathbf{D}_4 \boxplus \mathbf{E}_6 \boxplus \mathbf{E}_7$

Filling set $S = <s_1,\ s_2,\ s_3,\ s_4,\ s_5>$, where

$$s_1 = (0,\ 3,\ 3;\ 1;\ 0;\ 0), \quad s_2 = (3,\ 0,\ 3;\ 2;\ 0;\ 0),$$
$$s_3 = (3,\ 3,\ 3;\ 0;\ 0;\ 1), \quad s_4 = (2,\ 2,\ 0;\ 0;\ 1;\ 0),$$
$$s_5 = (2,\ 4,\ 2;\ 0;\ 0;\ 0).$$

Polynomial

$$P(x, y, z, t) = 1 + 2x^3 + 3x^2y + 6x^3y + 6x^2z + 6x^2yz + 12x^3yz + 3x^3t + 3xyt + 6x^3yt + 6x^3zt + 12x^2yzt + 6x^3yzt.$$

(20) $2\mathbf{A}_5 \boxplus \mathbf{D}_{10} \boxplus 2\mathbf{E}_6$

Filling set

$$S = <(3,\ 0;\ 1;\ 0,\ 0),\ (0,\ 3;\ 3;\ 0,\ 0),\ (2,\ 2;\ 0;\ 1,\ 0),\ (2,\ 4;\ 0;\ 0,\ 1)>.$$

Polynomial

$$P(x, y, z) = 1 + 2xy + x^2y + 4x^2z + 12x^2yz + 4xz^2 + 4xyz^2 + 8x^2yz^2.$$

(21) $\mathbf{A}_5 \boxplus \mathbf{A}_{11} \boxplus \mathbf{D}_9 \boxplus \mathbf{E}_7$

Filling set

$$S = <(0; 3; 1; 1), (3; 6; 0; 1), (2; 4; 0; 0)>.$$

Polynomial

$$P(x, y, z, t) = 1 + 2xy + yz + 8xyz + 3xyt + xzt + 2yzt + 6xyzt.$$

(22) $\mathbf{2A}_7 \boxplus \mathbf{2D}_5 \boxplus \mathbf{D}_8$

Filling set $S = <(1, 1; 1, 0; 2), (2, 0; 1, 1; 0), (0, 0; 2, 2; 1)>$.
Polynomial

$$P(x, y, z) = 1 + x^2 + 4x^2y + 6xy^2 + 4x^2y^2 + 2xz + 20x^2yz + y^2z + 4xy^2z + 21x^2y^2z.$$

(23) $\mathbf{2A}_7 \boxplus \mathbf{D}_5 \boxplus \mathbf{D}_{13}$

Filling set $S = <(1, 3; 1; 0), (2, 0; 1, 1)>$.
Polynomial $P(x, y, z) = 1 + x^2 + 6x^2y + 6x^2z + 6xyz + 12x^2yz$.

(24) $\mathbf{2A}_7 \boxplus \mathbf{2D}_9$

Filling set $S = <(1, 1; 1, 0), (2, 0; 1, 1)>$.
Polynomial $P(x, y) = 1 + x^2 + 12x^2y + 6xy^2 + 12x^2y^2$.

(25) $\mathbf{2A}_9 \boxplus \mathbf{D}_{14}$

Filling set $S = <(5, 0; 1), (0, 5; 3), (2, 4; 0)>$.
Polynomial $P(x, y) = 1 + 4x^2 + 2xy + 13x^2y$.

(26) $\mathbf{2A}_9 \boxplus \mathbf{2E}_7$

Filling set $S = <(5, 0; 1, 0), (0, 5; 0, 1), (2, 4; 0, 0)>$.
Weight polynomial $P(x, y) = 1 + 4x^2 + 2xy + 8x^2y + 5x^2y^2$.

(27) $\mathbf{A}_{11} \boxplus \mathbf{D}_{15} \boxplus \mathbf{E}_6$

Filling set $S = <(3; 1; 0), (4; 0; 1)>$.
Polynomial $P(x, y, z) = 1 + 3xy + 2xz + 6xyz$.

(28) $\mathbf{A}_{15} \boxplus \mathbf{D}_5 \boxplus \mathbf{D}_{12}$

Filling set $S = \langle (2;\ 1;\ 1),\ (0;\ 2;\ 3) \rangle$.
Polynomial $P(x, y, z) = 1 + x + 2xy + 2xz + yz + 9xyz$.

(29) $\mathbf{A}_{15} \boxplus \mathbf{D}_{17}$

Filling set $S = \langle (2;\ 1) \rangle$.
Polynomial $P(x, y) = 1 + x + 6xy$.

3. Root systems with deficiency 12

There are 10 root systems of rank 32 and deficiency 12 appearing as the root system of a (unique) even unimodular lattice of rank 32.

(1) $4\mathbf{A}_1 \boxplus 4\mathbf{A}_7$

The filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$ is given by

$$s_1 = (1,\ 0,\ 0,\ 0;\ 1,\ 1,\ 1,\ 1), \qquad s_2 = (1,\ 1,\ 0,\ 0;\ 2,\ 2,\ 0,\ 0),$$
$$s_3 = (0,\ 1,\ 1,\ 0;\ 0,\ 2,\ 6,\ 0), \qquad s_4 = (1,\ 1,\ 1,\ 1;\ 0,\ 0,\ 0,\ 4).$$

The weight enumerator polynomial of the corresponding metabolizer reads

$$P(x, y) = 1 + 4x^4y + 6y^2 + 24x^2y^2 + 48x^2y^3 + 4x^4y^3 + 9y^4 + 64xy^4 + 24x^2y^4 + 64x^3y^4 + 8x^4y^4.$$

(2) $4\mathbf{A}_2 \boxplus 4\mathbf{A}_5 \boxplus \mathbf{D}_4$

Filling set $S = \langle s_1, s_2, s_3 \rangle \times \langle s_4, s_5, s_6, s_7 \rangle$, where

$$s_1 = (0,\ 0,\ 0,\ 0;\ 3,\ 3,\ 3,\ 3;\ 0), \quad s_2 = (0,\ 0,\ 0,\ 0;\ 3,\ 3,\ 0,\ 0;\ 1),$$
$$s_3 = (0,\ 0,\ 0,\ 0;\ 0,\ 3,\ 3,\ 0;\ 2),$$
$$s_4 = (1,\ 1,\ 1,\ 1;\ 2,\ 0,\ 0,\ 0;\ 0),$$
$$s_5 = (1,\ -1,\ 1,\ -1;\ 0,\ 2,\ 0,\ 0;\ 0),$$
$$s_6 = (1,\ 1,\ -1,\ -1;\ 0,\ 0,\ 2,\ 0;\ 0),$$
$$s_7 = (1,\ -1,\ -1,\ 1;\ 0,\ 0,\ 0,\ 2;\ 0).$$

Weight enumerator polynomial

$$P(x, y, z) = 1 + 8x^4y + 24x^2y^2 + 32x^3y^3 + y^4 + 16xy^4 + 24x^2y^4 + 32x^3y^4 + 24x^4y^4 + 6y^2z + 24x^2y^2z + 24x^4y^2z + 96x^2y^3z + 96x^3y^3z + 24x^4y^3z + 48xy^4z + 24x^2y^4z + 96x^3y^4z + 48x^4y^4z.$$

(3) $$\mathbf{4A_2 \boxplus 4E_6}$$

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$\begin{aligned} s_1 &= (1,0,0,0;1, \quad 1, \quad 1, \quad 1),\\ s_2 &= (0,1,0,0;1, -1, \quad 1, -1),\\ s_3 &= (0,0,1,0;1, \quad 1, -1, -1),\\ s_4 &= (0,0,0,1;1, -1, -1, \quad 1). \end{aligned}$$

Weight enumerator polynomial

$$P(x, y) = 1 + 8x^4y + 24x^2y^2 + 32x^3y^3 + 8xy^4 + 8x^4y^4.$$

(4) $$\mathbf{2A_2 \boxplus 2A_{11} \boxplus D_6}$$

Filling set

$$S = \langle (0,\ 0;\ 3,\ 3;\ 1),\ (0,\ 0;\ 6,\ 0;\ 2),\ (1,\ 1,\ 4,\ 0;\ 0),\ (1,\ 2;\ 0,\ 4;\ 0) \rangle.$$

Polynomial

$$\begin{aligned} P(x, y, z) = 1 &+ 4x^2y + y^2 + 8xy^2 + 4x^2y^2 + 2yz + 4x^2yz + 4y^2z\\ &+ 24xy^2z + 20x^2y^2z. \end{aligned}$$

(5) $$\mathbf{A_2 \boxplus A_9 \boxplus A_{14} \boxplus E_7}$$

Filling set $S = \langle (1;\ 0;\ 5;\ 0),\ (0;\ 2;\ 3;\ 0),\ (0;\ 5;\ 0;\ 1) \rangle$.
Weight polynomial

$$P(x, y, z, t) = 1 + 2xz + 4yz + 8xyz + yt + 4yzt + 10xyzt.$$

(6) $$\mathbf{A_2 \boxplus A_{23} \boxplus E_7}$$

Filling set $S = \langle (1;\ 8;\ 0),\ (0;\ 6;\ 1) \rangle$.
Weight enumerator polynomial

$$P(x, y, z) = 1 + y + 4xy + 2yz + 4xyz.$$

(7) $$\mathbf{6A_3 \boxplus 2A_7}$$

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$\begin{aligned} s_1 &= (2,\ 1,\ 1,\ 1,\ 1,\ 0;\ 0,\ 0), & s_2 &= (1,\ 2,\ 1,\ 3,\ 0,\ 1;\ 0,\ 0),\\ s_3 &= (1,\ 1,\ 1,\ 0,\ 0,\ 0;\ 1,\ 1), & s_4 &= (0,\ 2,\ 1,\ 1,\ 0,\ 0;\ 2,\ 0). \end{aligned}$$

Weight enumerator polynomial

$$\begin{aligned} P(x, y) = 1 &+ 3x^4 + 12x^5 + 6x^2y + 24x^3y + 48x^5y + 18x^6y + y^2\\ &+ 72x^3y^2 + 123x^4y^2 + 132x^5y^2 + 72x^6y^2. \end{aligned}$$

(8) $\mathbf{2A_4 \boxplus 2A_9 \boxplus D_6}$

Filling set

$$S = <(0, 0; 5, 0; 1), \ (0, 0; 0, 5; 3), \ (1, 0; 2, 2; 0), \ (0, 1; 2, 8; 0)>.$$

Polynomial

$$P(x, y, z) = 1 + 8x^2y + 8xy^2 + 8x^2y^2 + 2yz + 8x^2yz + y^2z + 24xy^2z + 40x^2y^2z.$$

(9) $\mathbf{A_4 \boxplus A_{19} \boxplus D_9}$

Filling set $S = <(0; 5; 1), \ (1; 4; 0)>$.
Polynomial $P(x, y, z) = 1 + 4xy + 3yz + 12xyz$.

(10) $\mathbf{A_8 \boxplus A_{17} \boxplus E_7}$

Filling set $S = <(4; 2; 0), \ (0; 9; 1)>$.
Weight polynomial $P(x, y, z) = 1 + 8xy + yz + 8xyz$.

4. Root systems of deficiency 14

There are 5 root systems with deficiency 14 which appear as a complete root system in an even unimodular lattice of rank 32. There is only one lattice for each realizable root system.

(1) $\mathbf{2A_1 \boxplus 2A_{15}}$

Filling set

$$S = <(1, 0; 2, 2), \ (1, 1; 4, 0)>.$$

The weight enumerator polynomial is

$$P(x, y) = 1 + 2y + 4x^2y + 5y^2 + 16xy^2 + 4x^2y^2.$$

(2) $\mathbf{10A_2 \boxplus 2E_6}$

A filling set $S = <s_1, s_2, s_3, s_4, s_5, s_6>$ is as follows

$$\begin{aligned}
s_1 &= (0, 1, 1, 1, 1, 1, 1, 1, 1, 1; 0, 0),\\
s_2 &= (1, 1, 0, 1, 1, 2, 2, 2, 2, 1; 0, 0),\\
s_3 &= (1, 2, 1, 2, 0, 1, 1, 2, 2, 1; 0, 0),\\
s_4 &= (1, 1, 2, 2, 1, 1, 0, 2, 1, 2; 0, 0),\\
s_5 &= (1, 1, 1, 1, 0, 0, 0, 0, 0, 0; 1, 0),\\
s_6 &= (0, 0, 1, 2, 2, 1, 0, 0, 0, 0; 0, 1).
\end{aligned}$$

The weight enumerator of the corresponding metabolizer is

$$P(x, y) = 1 + 60x^6 + 20x^9 + 60x^4y + 240x^7y + 24x^{10}y + 144x^5y^2 + 180x^8y^2.$$

See the following Section 7 for the relationship of this root system with conference matrices.

(3) $$\mathbf{2A_2 \boxplus 2A_3 \boxplus 2A_{11}}$$

Filling set

$$S = <(0,0;1,2;3,0),\ (0,0;2,1;0,3),\ (1,1;0,0;4,0),\ (1,2;0,0;0,4)>.$$

Polynomial

$$P(x, y, z) = 1 + 4x^2z + 2yz + 4x^2yz + 4y^2z + 8x^2y^2z + 4xz^2 + 4yz^2 + 24xyz^2 + 20x^2yz^2 + 5y^2z^2 + 36xy^2z^2 + 28x^2y^2z^2.$$

(4) $$\mathbf{2A_5 \boxplus 2A_{11}}$$

Filling set

$$S = <(3,\ 0;\ 3,\ 3),\ (3,\ 3;\ 6,\ 0),\ (2,\ 0;\ 4,\ 0),\ (0,\ 2;\ 0,\ 4)>.$$

Polynomial

$$P(x, y) = 1 + 4xy + 6x^2y + y^2 + 16xy^2 + 44x^2y^2.$$

(5) $$\mathbf{A_{11} \boxplus A_{15} \boxplus E_6}$$

Filling set $S = <(3;\ 2;\ 0),\ (4;\ 0;\ 1)>$.
Polynomial $P(x, y, z) = 1 + y + 6xy + 2xz + 14xyz$.

5. Root systems of deficiency 15

There are 8 root systems of deficiency 15 which occur as the complete root system of an even unimodular lattice of rank 32. Each lattice is uniquely determined by its root system.

(1) $$\mathbf{A_1 \boxplus 3A_6 \boxplus A_{13}}$$

Filling set

$$S = <(1;\ 0,\ 0,\ 0;\ 7),\ (0;\ 1,\ 2,\ 3;\ 0),\ (0;\ 2,\ 6,\ 0;\ 2)>.$$

Polynomial

$$P(x, y, z) = 1 + 6y^3 + xz + 18y^2z + 18xy^2z + 24y^3z + 30xy^3z.$$

Here again, the polynomial is the only candidate satisfying duality. In turn, the given filling set is uniquely determined by the polynomial.

(2) $$\mathbf{A}_1 \boxplus \mathbf{A}_{10} \boxplus \mathbf{A}_{21}$$

Filling set $S = \langle (1;\ 0;\ 11),\ (0;\ 1;\ 8) \rangle$.
Polynomial $P(x, y, z) = 1 + xz + 10yz + 10xyz$.

(3) $$\mathbf{A}_1 \boxplus \mathbf{A}_{31}$$

Filling set $S = \langle (1;\ 4) \rangle$.
Polynomial $P(x, y) = 1 + 3y + 4xy$.

(4) $$13\mathbf{A}_2 \boxplus \mathbf{E}_6$$

Filling set $S = \langle s_0, s_1, s_2, s_3, s_4, s_5, s_6 \rangle$ as follows

$$\begin{aligned}
s_0 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1;\ 1),\\
s_1 &= (2, 0, 1, 0, 2, 1, 2, 1, 0, 0, 0, 0, 0;\ 0),\\
s_2 &= (0, 2, 0, 1, 0, 2, 1, 2, 1, 0, 0, 0, 0;\ 0),\\
s_3 &= (0, 0, 2, 0, 1, 0, 2, 1, 2, 1, 0, 0, 0;\ 0),\\
s_4 &= (0, 0, 0, 2, 0, 1, 0, 2, 1, 2, 1, 0, 0;\ 0),\\
s_5 &= (0, 0, 0, 0, 2, 0, 1, 0, 2, 1, 2, 1, 0;\ 0),\\
s_6 &= (0, 0, 0, 0, 0, 2, 0, 1, 0, 2, 1, 2, 1;\ 0).
\end{aligned}$$

The weight enumerator is

$$\begin{aligned}
P(x, y) = {} & 1 + 156x^6 + 494x^9 + 78x^{12} + 26x^4y + 624x^7y\\
& + 780x^{10}y + 28x^{13}y.
\end{aligned}$$

Note that $M_0 = M \cap T(13\mathbf{A}_2)$, where M is the metabolizer generated by S in $T(13\mathbf{A}_2 \boxplus \mathbf{E}_6)$, is the cyclic code in $\mathbf{F}_3[\mathbf{x}]/(\mathbf{x}^{13} - \mathbf{1})$ generated by

$$\begin{aligned}
g(x) &= x^7 - x^6 + x^5 - x^4 + x^2 - 1\\
&= (x-1)(x^3 + x^2 - 1)(x^3 - x^2 - x - 1),
\end{aligned}$$

with roots $\alpha^4, \alpha^7, \alpha^8, \alpha^{10}, \alpha^{11}, \alpha^{12}, \alpha^{13} = 1$, where α is a root of $X^3 - X - 1$ in $\mathbf{F}_{27}$.

(5) $$\mathbf{A}_2 \boxplus \mathbf{A}_5 \boxplus \mathbf{A}_8 \boxplus \mathbf{A}_{17}$$

Filling set

$$S = \langle (0;\ 3;\ 0;\ 9),\ (1;\ 4;\ 1;\ 4),\ (1;\ 2;\ 3;\ 0) \rangle.$$

Polynomial

$P(x, y, z, t) = 1 + 2xyz + yt + 4xyt + 2zt + 6xzt + 14yzt + 24xyzt.$

Here, in order to prove uniqueness, one should first observe that the weight enumerator of the metabolizer is uniquely determined by the duality theorem of Section 4. It is then easy to see that the above filling set is the only possible one.

(6) $$\mathbf{A}_2 \boxplus 3\mathbf{A}_8 \boxplus \mathbf{E}_6$$

Filling set

$$S = <(0;\ 1,\ 1,\ 1;\ 1),\ (1;\ 3,\ 0,\ 0;\ 1),\ (1;\ 0,\ 3,\ 0;\ 1)>.$$

Weight enumerator

$$P(x, y, z) = 1 + 6y^2 + 2y^3 + 18xy^3 + 6xyz + 6xy^2z + 18y^3z + 24xy^3z.$$

For the proof of uniqueness, one first observes that the above polynomial is the only one compatible with the requirement of duality. Then, the only 6 candidates for the weight xyz are $\pm(1;3,0,0;1)$, $\pm(1;0,3,0;1)$ and $\pm(1;0,0,3;1)$.

The vector $(0;1,1,1;1)$ is then uniquely determined, up to obvious automorphisms, by the requirement of compatibility with the other 3 vectors.

(7) $$\mathbf{A}_6 \boxplus \mathbf{A}_{20} \boxplus \mathbf{E}_6$$

Filling set $S = <(0;\ 7;\ 1),\ (2;\ 3;\ 0)>$.
Polynomial $P(x, y, z) = 1 + 6xy + 2yz + 12xyz$.

(8) $$\mathbf{A}_{26} \boxplus \mathbf{E}_6$$

Filling set $S = <(3;\ 1)>$.
Weight enumerator $P(x, y) = 1 + 2x + 6xy$.

6. Root systems of deficiency 16

There are 5 root systems of deficiency 16 occuring as the root system of even unimodular lattices of rank 32. Each of these lattices is determined by its root system.

(1) $$16\mathbf{A}_2$$

The system of filling vectors can be taken as the rows of an 8×16 matrix

$$S = (I, H)\ ,$$

where I is the 8×8 identity matrix and H is the Hadamard matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

The weight enumerator is

$P(x) = 1 + 224x^6 + 2720x^9 + 3360x^{12} + 256x^{15}$.

The uniqueness of the lattice with this root system follows from the classification of self-dual codes in $\mathbf{F}_3^{16}$ due to J. Conway, V. Pless and N. Sloane in [CPS].

(2) $\mathbf{2A_2 \boxplus 2A_{14}}$

Filling set $S = \langle (1, 0; 5, 0),\ (0, 1; 0, 5),\ (0, 0; 3, 6) \rangle$.
Weight enumerator $P(x, y) = 1 + 4xy + 4y^2 + 16xy^2 + 20x^2y^2$.

(3) $\mathbf{8A_4}$

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where s_1, s_2, s_3, s_4 can be taken to be the rows of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

The weight enumerator is

$P(x) = 1 + 48x^4 + 32x^5 + 288x^6 + 128x^7 + 128x^8$.

For the proof of uniqueness, see the comments in the next section.

(4) $\mathbf{4A_8}$

Filling set $S = \langle (1, 1, 4, 0),\ (1, -1, 0, 4) \rangle$.
Weight enumerator $P(x) = 1 + 32x^3 + 48x^4$.

(5) $\mathbf{2A_{16}}$

Filling set $S = \langle (1, 4) \rangle$.
Weight enumerator $P(x) = 1 + 16x^2$.

7. Comments

In this section we give some details on the construction and on the proof of uniqueness of the even unimodular lattices of rank 32 with root systems $\mathbf{8A_1} \boxplus \mathbf{8A_3}$, $\mathbf{10A_2} \boxplus \mathbf{2E_6}$, $\mathbf{13A_2} \boxplus \mathbf{E_6}$, and $\mathbf{8A_4}$.

The first example, $\mathbf{8A_1} \boxplus \mathbf{8A_3}$, involves a rather heavy analysis, requiring some overview of the self-orthogonal codes in $T(\mathbf{8A_3})$ which is also necessary in order to treat the other root systems containing $\mathbf{8A_3}$.

The last three examples are hopefully more attractive.

(1) $\mathbf{8A_1} \boxplus \mathbf{8A_3}$

Here we have deficiency 8 and any metabolizer M must be of order 2^{12}.

If M is an admissible metabolizer and $P = P(x, y)$ its weight enumerator polynomial, the duality theorem of Section 4 provides an underdetermined linear system for the coefficients of P. The coefficients c, α, β, γ of x^6y^8, x^8y^6, x^8y^7 and x^8y^8 respectively can be taken as parameters and all other coefficients are then linear expressions in c, α, β, γ.

Let the polynomial P be

$$P(x, y) = 1 + c_1y^4 + c_2y^5 + c_3y^6 + c_4y^7 + c_5y^8 + \dots ,$$

where the dots stand for the terms which are divisible by x.

Then, the coefficients $c_1, \dots, c_5$ satisfy the equations

$$\begin{aligned} c_1 &= -37 + \alpha + 2\beta + 3\gamma, \\ c_2 &= 68 - 2\alpha - 3\beta - 4\gamma, \\ c_3 &= \alpha, \\ c_4 &= \beta, \\ c_5 &= \gamma. \end{aligned}$$

This shows that $1 + c_1 + c_2 + c_3 + c_4 + c_5 = 32$. If $M \subset T(\mathbf{8A_1} \boxplus \mathbf{8A_3})$ is an admissible metabolizer, then $1 + c_1y^4 + c_2y^5 + c_3y^6 + c_4y^7 + c_5y^8$ can be interpreted as the weight enumerator of $N = M \cap T(\mathbf{8A_3})$. Thus $|N| = 32$.

Step 1. We will first show that N is uniquely determined up to a (norm preserving) automorphism of $T(\mathbf{8A_3})$.

Let $N' = N \cap 2T(\mathbf{8A_3})$. Consider the exact sequence

$$0 \to N' \to N \xrightarrow{\pi} N'' \to 0 ,$$

where π is the restriction to N of the projection $T(\mathbf{8A}_3) \to T(\mathbf{8A}_3)/2T(\mathbf{8A}_3)$, and $N'' = \pi(N) \subset T(\mathbf{8A}_3)/2T(\mathbf{8A}_3)$.

The map $\psi : N'' \to N'$ given by $\psi(x) = 2y$, where $\pi(y) = x$ is well defined, linear and injective. Hence, $|N''| \leqslant |N'|$ and since $|N| = |N'| \cdot |N''|$, it follows that there are 2 cases to be examined:

(1) $$|N'| = 16 \quad \text{and} \quad |N''| = 2\,,$$

(2) $$|N'| = 8 \quad \text{and} \quad |N''| = 4\,,$$

In case (1), there is just one possibility for N', namely

$$N' = \langle (2,2,2,2,0,0,0,0),\ (2,2,0,0,2,2,0,0),\ (2,2,0,0,0,0,2,2),\ (2,0,2,0,2,0,2,0) \rangle$$

and there are 2 corresponding possibilities for N, depending on whether $\psi(N'') = \langle (2,2,2,2,2,2,2,2) \rangle$ or $\psi(N'') = \langle (2,2,2,2,0,0,0,0) \rangle$. Note that there is a single orbit of vectors of weight 4 under the group of permutations of the 8 coordinates in $T(\mathbf{8A}_3)$ preserving N'.

The 2 cases are specified by $N = N_1$ or N_2, where

$$N_1 = \langle (1,1,1,1,1,1,1,1),\ (2,2,2,2,0,0,0,0),\ (2,2,0,0,2,2,0,0),\ (2,0,2,0,2,0,2,0) \rangle,$$

and

$$N_2 = \langle (1,1,1,1,2,0,0,0),\ (2,2,0,0,2,2,0,0),\ (2,2,0,0,0,0,2,2),\ (2,0,2,0,2,0,2,0) \rangle.$$

For N_1, the weight polynomial is

$$P_1(0,y) = 1 + 14y^4 + 17y^8\,.$$

For N_2, the weight polynomial is

$$P_2(0,y) = 1 + 14y^4 + 8y^5 + 8y^7 + y^8\,.$$

However, in the second case, the polynomial coefficients of $P_2(0,y)$ would imply

$$\alpha = 0, \qquad \beta = 8, \qquad \gamma = 1$$

and thus $c_1 = -18$ for the coefficient of y^4 in $P(x,y)$. This case is therefore impossible and we retain only the possibility $N = N_1$ and

$$P_N(0,y) = 1 + 14y^4 + 17y^8\,.$$

As we shall see, it will actually turn out that the above subgroup N_1 is the only acceptable choice for $N = M \cap T(\mathbf{8A}_3)$.

In case (2), i.e. $|N'| = 8$, $|N''| = 4$, the possibilities for the weight polynomial of N' are

(2.1) $$P_{N'} = 1 + 5y^4 + 2y^6, \text{ or}$$

(2.2) $$P_{N'} = 1 + 6y^4 + y^8, \quad \text{or}$$

(2.3) $$P_{N'} = 1 + 7y^4 .$$

Moreover, in each case, N' is unique up to permutation of coordinates:

(2.1) $N' = < (2,2,2,2,2,2,0,0), (0,0,2,2,2,2,2,2), (2,0,2,0,2,0,2,0) >$,

(2.2) $N' = < (2,2,2,2,0,0,0,0), (0,0,0,0,2,2,2,2), (2,2,0,0,2,2,0,0) >$,

(2.3) $N' = < (2,2,2,2,0,0,0,0), (2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0) >$.

In these cases, the image of $\psi : N'' \to N'$ is a plane i.e. $|\psi(N'')| = 4$ and since the admissible vectors of weight 6 in $T(\mathbf{8A}_3)$ are not divisible by 2 in the set of admissible vectors, it follows that $\psi(N'')$ contains only vectors of weight 0, 4 or 8.

In case (2.1), there is just one orbit of planes with all non-zero vectors of weight 4 under the action of the group of permutation of coordinates preserving N', namely the orbit of $< (2,2,0,0,0,0,2,2), (2,2,0,2,0,2,0,2,0) >$. However, it is easy to see that none of the admissible vectors $v \in T(\mathbf{8A}_3)$ such that $2v = (2,0,2,0,2,0,2,0)$, is compatible with N'. Typically, if $v = (1,2,1,0,1,0,1,0)$, then $v + (2,2,2,2,2,2,0,0) = (3,0,3,2,3,2,1,0)$ which has norm 5 and therefore is not admissible. Thus, in fact, case (2.1) cannot occur.

In case (2.2), where

$$N' = < (2,2,2,2,2,2,2,2), (2,2,2,2,0,0,0,0), (2,2,0,0,2,2,0,0) >$$

there are 2 orbits of planes in N' under the action of the automorphism group of N':

— The orbit $[u_1, u_2]$, $[u_1, u_3]$, $[u_1, u_2 + u_3]$ consisting of the planes containing $u_1 = (2,2,2,2,2,2,2,2)$ which is fixed by every automorphism.

— The orbit consisting of the planes $[u_2, u_3]$, $[u_1 + u_2, u_3]$, $[u_2, u_1 + u_3]$, $[u_1 + u_2, u_1 + u_3]$ not containing u_1.

Here, we have set $u_2 = (2,2,2,2,0,0,0,0)$ and $u_3 = (2,2,0,0,2,2,0,0)$.

Thus, we have two possible choices for the plane $\psi(N'')$, namely $[u_1, u_2]$ or $[u_2, u_3]$.

If $\psi(N'') = [u_1, u_2]$ is chosen, an enumeration of the possibilities shows that we can then assume N to be of the form

$$N = < (1,1,1,1,1,1,1,1), (1,1,1,3,0,2,0,0), (2,2,0,0,2,2,0,0) > .$$

The resulting weight polynomial for N, namely

$$P_N = 1 + 6y^4 + 8y^5 + 8y^7 + 9y^8$$

determines the coefficients α, β, γ as

$$\alpha = 0, \quad \beta = 8, \quad \gamma = 9 ,$$

and then, throwing in the monomials containing x, P_M becomes

$$\begin{aligned} P_M(x,y) = 1 &+ 6y^4 + 8y^5 + 8y^7 + 9y^8 + 24x^2y^3 + cx^2y^4 \\ &+ (400-4c)x^2y^5 + 6cx^2y^6 + (472-4c)x^2y^7 + cx^2y^8 \\ &+ 32x^4y^2 + (344-2c)x^4y^4 + (112+8c)x^4y^5 \\ &+ (1232-12c)x^4y^6 + (112+8c)x^4y^7 + (408+2c)x^4y^8 \\ &+ 24x^6y^3 + cx^8y^4 + 8x^8y^5 + 8x^8y^7 + 9x^8y^8, \end{aligned}$$

where c still has to be determined.

In order to calculate c, we examine the possible vectors of weight x^2y^7 in M. It is easy to see, considering the norm, that the only candidates must have the form $(1,1,0,0,0,0,0,0;2,2,2,2,2,2,2,0)$ up to permutation of coordinates. But it is immediate that any such vector fails to be compatible with the vector $(0,0,0,0,0,0,0,0;2,2,2,2,2,2,2,2) \in N \subset M$ because their sum would have norm 2. Therefore, the coefficient of x^2y^7 in P_M must be 0.

This forces $c = 118$. Unfortunately, the coefficient of x^2y^5 then becomes negative. Hence, there is no admissible metabolizer with this choice of $N = M \cap T(\mathbf{8A}_3)$.

The other choice (still under case (2.2)) is $\psi(N'') = [u_2, u_3]$. Here, an examination of the possible choices for N leads to either

$$N = \langle (1,1,1,1,2,0,0,0), (1,1,2,2,1,1,0,2), (0,0,0,0,2,2,2,2) \rangle ,$$

or

$$N = \langle (1,1,1,1,2,0,0,0), (1,3,0,2,1,1,0,0), (0,0,0,0,2,2,2,2) \rangle .$$

In both cases, the weight polynomial for N is

$$P_N = 1 + 6y^4 + 12y^5 + 12y^7 + y^8 ,$$

and this determines the parameters $\alpha = 0$, $\beta = 12$, $\gamma = 1$, contradicting the equation $c_1 = -37 + \alpha + 2\beta + 3\gamma$.

There remains the case (2.3), where

$$N' = \langle (2,2,2,2,0,0,0,0), (2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0) \rangle .$$

In this case, it is easy to see that there is just one orbit of planes in N' under the action of the group of coordinate permutations preserving N'. Hence, we may assume $\psi(N'') = [u_1, u_2]$, where $u_1 = (2,2,2,2,0,0,0,0)$ and $u_2 = (2,2,0,0,2,2,0,0)$ and there are 4 choices for N:

They are $< N_i, u_3 >$, $i = 1,2,3,4$, where $u_3 = (2,0,2,0,2,0,2,0)$ and

$$\begin{aligned}
N_1 &= <(1,1,1,1,2,0,0,0), (1,1,0,0,1,1,2,0)>,\\
N_2 &= <(1,1,1,1,2,0,0,0), (1,1,0,0,1,1,0,2)>,\\
N_3 &= <(1,1,1,1,0,0,0,2), (1,1,2,0,1,1,0,0)>,\\
N_4 &= <(1,1,1,1,0,0,0,2), (1,1,2,0,1,1,2,2)>.
\end{aligned}$$

The resulting polynomials P_N are $1 + 7y^4 + 18y^5 + 6y^7$ in the first case, and $1 + 7y^4 + 10y^5 + 14y^7$ in the last 3 cases.

In both instances, the values of the parameters α, β, γ contradict the equation for c_1.

Summarizing this first phase of the analysis, we necessarily have

$$\begin{aligned}
N = <&(1,1,1,1,1,1,1,1), (2,2,2,2,0,0,0,0),\\
&(2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0)>,
\end{aligned}$$

and the vanishing of the coefficient of x^2y^7 (because any vector of weight x^2y^7 is incompatible with $(0,0,0,0,0,0,0,0;2,2,2,2,2,2,2,2) \in N$) forces the weight polynomial to be as announced:

$$\begin{aligned}
P(x,y) = 1 &+ x^8 + 56x^4y^2 + 14y^4 + 112x^2y^4 + 112x^4y^4 + 112x^6y^4\\
&+ 14x^8y^4 + 896x^4y^5 + 672x^2y^6 + 56x^4y^6 + 672x^6y^6\\
&+ 896x^4y^7 + 17y^8 + 112x^2y^8 + 224x^4y^8 + 112x^6y^8 + 17x^8y^8.
\end{aligned}$$

Thus the weight enumerator of any putative admissible metabolizer is uniquely determined after all, and more importantly $N = M \cap T(\mathbf{8A}_3)$ is uniquely determined as

$$\begin{aligned}
N = <&(1,1,1,1,1,1,1,1), (2,2,2,2,0,0,0,0),\\
&(2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0)>.
\end{aligned}$$

STEP 2. Now, since $|M| = 2^{12}$ and $|N| = 2^5$ the projection of any metabolizer M into $T(\mathbf{8A}_1)$ must be a 7-dimensional subspace. Since the polynomial P_M contains only monomials with x to an even power, the projection of M into $T(\mathbf{8A}_1)$ consists exactly of the vectors of even weight. Let $e_i \in T(\mathbf{8A}_1) = \mathbf{F}_2^8$ be the vectors with coordinates i and $i+1$ equal to 1 and all others $0 (i = 1, \ldots, 7)$. If $v \in T(\mathbf{8A}_3)$, we use the (hopefully) self-explanatory notation $e_i + v \in T(\mathbf{8A}_1) \boxplus T(\mathbf{8A}_3)$. Obviously, M admits a system of generators consisting of vectors of the form $e_k + \upsilon_{i_k}, k = 1, \ldots, 7$ together with N.

There is a list of 28 classes $\upsilon + N$ modulo N of vectors υ such that $e_i + \upsilon$ is compatible with N, i.e. such that the subgroup of $T(\mathbf{8A}_1) \boxplus T(\mathbf{8A}_3)$ generated by $e_i + \upsilon$ and N consists entirely of admissible vectors.

Each class has a representative with all non-zero coordinates equal to 1 or 3 and first non-zero coordinate equal to 1. The list reads as follows:

$$\begin{array}{ll}
\upsilon_0 = (0,0,0,0,1,1,1,1), & \upsilon_7 = (0,0,0,0,1,1,3,3),\\
\upsilon_1 = (0,0,1,1,1,1,0,0), & \upsilon_8 = (0,0,1,1,3,3,0,0),\\
\upsilon_2 = (0,0,1,1,0,0,1,1), & \upsilon_9 = (0,0,1,1,0,0,3,3),\\
\upsilon_3 = (0,1,0,1,0,1,0,1), & \upsilon_{10} = (0,1,0,1,0,3,0,3),\\
\upsilon_4 = (0,1,0,1,1,0,1,0), & \upsilon_{11} = (0,1,0,1,3,0,0,3),\\
\upsilon_5 = (0,1,1,0,1,0,0,1), & \upsilon_{12} = (0,1,1,0,3,0,0,3),\\
\upsilon_6 = (0,1,1,0,0,1,1,0), & \upsilon_{13} = (0,1,1,0,0,3,3,0),
\end{array}$$

$$\begin{array}{ll}
\upsilon_{14} = (0,0,0,0,1,3,1,3), & \upsilon_{21} = (0,0,0,0,1,3,3,1),\\
\upsilon_{15} = (0,0,1,3,1,3,0,0), & \upsilon_{22} = (0,0,1,3,3,1,0,0),\\
\upsilon_{16} = (0,0,1,3,0,0,1,3), & \upsilon_{23} = (0,0,1,3,0,0,3,1),\\
\upsilon_{17} = (0,1,0,3,0,1,0,3), & \upsilon_{24} = (0,1,0,3,0,3,0,1),\\
\upsilon_{18} = (0,1,0,3,1,0,3,0), & \upsilon_{25} = (0,1,0,3,3,0,1,0),\\
\upsilon_{19} = (0,1,3,0,1,0,0,3), & \upsilon_{26} = (0,1,3,0,3,0,0,1),\\
\upsilon_{20} = (0,1,3,0,0,1,3,0), & \upsilon_{27} = (0,1,3,0,0,3,1,0).
\end{array}$$

Thus any admissible metabolizer M is generated by $N \subset T(\mathbf{8A}_3) \subset T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$, where

$$\begin{aligned} N = \langle & (1,1,1,1,1,1,1,1), (2,2,2,2,0,0,0,0),\\ & (2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0) \rangle, \end{aligned}$$

and 7 vectors of the form

$$s_1 = e_1 + \upsilon_{k_1}, s_2 = e_2 + \upsilon_{k_2}, \ldots, s_7 = e_7 + \upsilon_{k_7},$$

where $\upsilon_{k_1}, \upsilon_{k_2}, \ldots, \upsilon_{k_7}$ are taken from the above list.

A septet $(k_1, \ldots, k_7)$ such that the subgroup $M = \langle s_1, \ldots, s_7 \rangle + N$ is an admissible metabolizer (i.e. consisting only of vectors of integral, even norm $\neq 2$) will be called an *admissible septet* and the corresponding metabolizer $\langle s_1, \ldots, s_7 \rangle + N$ will be denoted $M(i_1, \ldots, i_7)$.

In order to determine the admissible septets it is not necessary to handle the $\binom{28}{7} \times 7! = 5967561600$ cases. One first makes a list P_0 of pairs (i, j)

such that

$$M_{i,j} = \langle e_1 + v_i, e_3 + v_j \rangle + N$$

is an admissible subgroup. The list P_0 contains 210 unordered pairs (420 if (i, j) and (j, i) are counted for 2).

The machine can then easily sort out the (unordered) quadruples (i, j, k, l) such that the 6 pairs $(i, j), (i, k), \ldots, (k, l)$ belong to P_0, a condition which is necessary for (i, j, k, l) to appear as $i = i_1, j = i_3, k = i_5, l = i_7$ in some admissible septet $(i_1, i_2, i_3, \ldots, i_7)$. A list Q of 105 quadruples comes out.

Note that if $(i_1, i_2, \ldots, i_7)$ is an admissible septet and (i'_1, i'_3, i'_5, i'_7) is any permutation of (i_1, i_3, i_5, i_7), there is a new triple (i'_2, i'_4, i'_6) such that $(i'_1, i'_2, i'_3, \ldots, i'_6, i'_7)$ is again an admissible septet and the corresponding metabolizers M, M' yield isomorphic lattices.

For instance, if $M = \langle e_1 + v_{i_1}, \ldots, e_7 + v_{i_7} \rangle + N$, then the permutation $\pi = (1\ 3)\ (2\ 4)$ on the first 8 coordinates (permuting the factors $T(\mathbf{A}_1)$) and leaving $T(\mathbf{8A}_3)$ fixed, carries M to

$$\begin{aligned} M' &= \langle e_3 + v_{i_1}, e_1 + e_2 + e_3 + v_{i_2}, e_1 + v_{i_3}, e_4 + v_{i_4}, \ldots, e_7 + v_{i_7} \rangle + N \\ &= \langle e_1 + v_{i_3}, e_2 + v'_{i_2}, e_3 + v_{i_1}, e_4 + v_{i_4}, \ldots, e_7 + v_{i_7} \rangle + N, \end{aligned}$$

where $v'_{i_2} = v_{i_1} + v_{i_2} + v_{i_3}$. Then, v'_{i_2} must be a vector $v_{i'_2}$ of the above basic list (up to addition of a vector of N). Therefore, $(i_3, i'_2, i_1, i_4, i_5, i_6, i_7)$ is an admissible septet. Thus, any equivalence class of admissible metabolizer can be represented by a septet $(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ such that $i_1 < i_3 < i_5 < i_7$.

Now, let G be the group of permutations of the last 8 coordinates in $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$ generated by

$$\alpha = (1\ 2)\ (3\ 4), \qquad \beta = (3\ 5)\ (4\ 6), \qquad \gamma = (1\ 7)\ (2\ 8), \qquad \rho = (1\ 6)\ (3\ 8)$$

permuting the 8 factors $T(\mathbf{A}_3)$ in $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$.

The group G has order 1344 and it operates on the set of classes *mod* N of the 28 vectors of the above basic list. It operates therefore also on the set Q of quadruples. The 105 quadruples forming Q are then divided into 3 orbits under this action, represented by the quadruples

$q_0 = (0, 7, 14, 21)$ with Gq_0 of cardinality 7,

$q_1 = (0, 7, 16, 23)$ with Gq_1 of cardinality 42,

$q_2 = (5, 10, 20, 25)$ with Gq_2 of cardinality 56.

Next, let P_1 be the set of pairs (i, j) such that

$$M'_{i,j} = \langle e_1 + v_i, e_2 + v_j \rangle + N$$

is an admissible subgroup of $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$, i.e. consisting entirely of vectors v such that the norm $\mathbf{n}(v)$ of v is an even integer $\neq 2$. The set P_1 contains 336 ordered pairs (obviously $(i, j) \in P_1$ implies $(j, i) \in P_1$). Any admissible septet $(i_1, \ldots, i_7)$ must be such that $(i_1, i_3, i_5, i_7) \in Q$, and $(i_k, i_{k+1}) \in P_1$ for $k = 1, \ldots, 6$, in addition to $(i_k, i_l) \in P_0$ for $|k - l| \geqslant 2$.

Given a quadruple $q = (i_1, i_3, i_5, i_7) \in Q$, it is not hard to sort out the set T_q of triples (i_2, i_4, i_6) such that $(i_1, i_2, \ldots, i_7)$ satisfies all the conditions on the pairs (i_k, i_l). We need to do this in fact only for the above 3 quadruples q_0, q_1, q_2, since any admissible septet can be carried by the action of G to a septet $(i_1, i_2, \ldots, i_7)$ completing q_0, q_1 or q_2 in the sense that $(i_1, i_3, i_5, i_7) = q_0, q_1$ or q_2.

It turns out that for each of these 3 quadruples $q = (i_1, i_3, i_5, i_7)$, there are 16 triples in the set T_q.

The resulting set of 48 septets can in fact still be reduced using the action of G. The subgroups of G fixing q_0, q_1 or q_2 are respectively of order 8, 4 and 1 and we are left with the following septets:

$$(0, 1, 7, 20, 14, 22, 21), \qquad (0, 1, 7, 20, 14, 23, 21)$$

completing q_0;

$$(0, 1, 7, 20, 16, 21, 23), \qquad (0, 1, 7, 20, 16, 22, 23)$$
$$(0, 9, 7, 20, 16, 21, 23), \qquad (0, 9, 7, 20, 16, 22, 23)$$

completing q_1;

and with the quadruple $q_2 = (5, 10, 20, 25)$ there are the 16 triples

(0, 14, 7),	(0, 14, 17),	(0, 19, 16),	(0, 19, 26),
(13, 14, 7),	(13, 14, 17),	(13, 19, 16),	(13, 19, 26),
(4, 11, 7),	(4, 11, 17),	(4, 8, 16),	(4, 8, 26),
(23, 11, 7),	(23, 11, 17),	(23, 8, 16),	(23, 8, 26),

forming the septets (5, 0, 10, 14, 20, 7, 25), etc.

Denote by $M(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ the subgroup

$$M(i_1, \ldots, i_7) = \langle e_1 + v_{i_1}, \ldots, e_7 + v_{i_7} \rangle + N.$$

We finish exploiting the operations of the permutation group S_8 acting on $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$ by permuting the first 8 coordinates.

It is easy to check that $\sigma_1 = (1\ 2) \in S_8$ acts on admissible metabolizers of the form $M(i_1, i_2, i_3, \ldots, i_7)$ by

$$\sigma_1 M(i_1, i_2, i_3, \ldots, i_7) = M(i_1, i_2', i_3, \ldots, i_7)\ ,$$

where $v_{i_2'}$ is the uniquely determined element in the basic list such that $v_{i_2'} \equiv v_{i_1} + v_{i_2}$ modulo N.

Similarly,

$$\sigma_k \, M(i_1, \ldots, i_7) = M(i_1', \ldots, i_7') ,$$

where $i_l' = i_l$ for $l \neq k - 1, k + 1$ and

$$v_{i_{k-1}'} \equiv v_{i_{k-1}} + v_{i_k} \text{ modulo } N, v_{i_{k+1}'} \equiv v_{i_k} + v_{i_{k+1}} \text{ modulo } N ,$$

for $k = 1, 2, \ldots, 6$;

$$\sigma_7 M(i_1, \ldots, i_7) = M(i_1, \ldots, i_5, i_6', i_7) ,$$

where $v_{i_6'} \equiv v_{i_6} + v_{i_7}$ modulo N.

Using $\sigma_1, \sigma_3, \sigma_5$ and σ_7 one first observes that all $M(i_1, i_2, \ldots, i_7)$ with the same quadruple $q = (i_1, i_3, i_5, i_7)$ are equivalent. Hence, the equivalence class of any admissible metabolizer is detected by its basic quadruple which can be q_0, q_1 or q_2. However, the permutation σ_6 carries $M(0, 1, 7, 20, 14, 22, 21)$ to $M(0, 1, 7, 20, 16, 22, 21)$. Similarly, the permutation $\pi = (7\,4\,5\,6\,3\,2\,1\,8)$ takes $M(5, 0, 10, 14, 20, 7, 25)$ to $M(0, 8, 7, 27, 14, 16, 21)$ which is equivalent to $M(0, 1, 7, 20, 14, 22, 21)$.

It is easy to let the machine verify that $M(0, 1, 7, 20, 14, 22, 21)$ actually is an admissible metabolizer and to pass from it to the filling set displayed in the table.

Thus, there is a single isomorphism class of 32-dimensional even, unimodular lattice with root system $\mathbf{8A_1} \boxplus \mathbf{8A_3}$.

(2) $\mathbf{10A_2} \boxplus \mathbf{2E_6}$

The only weight enumerator polynomial $P(x, y)$ for an admissible metabolizer in $T(\mathbf{10A_2} \boxplus \mathbf{2E_6})$ which is compatible with the duality theorem is

$$P(x, y) = 1 + 60x^6 + 20x^9 + 60x^4y + 240x^7y + 24x^{10}y + 144x^5y^2 + 180x^8y^2 .$$

Thus in $T(\mathbf{10A_2}) = \mathbf{F}_3^{10}$, the intersection $M_0 = M \cap T(\mathbf{10A_2})$ contains exactly 10 pairs $\{x, -x\}$ of vectors of Hamming weight 9.

Two distinct such pairs $\{x, -x\}$ and $\{x', -x'\}$ cannot have their vanishing coordinate at the same place. Indeed, suppose that for some i, we have $x_i' = x_i = 0$. Set $J = \{j \in \{1, \ldots, 10\} \mid x_j' = x_j \neq 0\}$ and $K = \{k \in \{1, \ldots, 10\} \mid x_k' = -x_k \neq 0\}$. Then $|J| + |K| = 9$, and $w(x + x') = |J|$, $w(x - x') = |K|$. The polynomial says that $|J| \neq 3$, $|K| \neq 3$. Hence the only possibility is $\{|J|, |K|\} = \{0, 9\}$ and $x' = \pm x$.

By numbering the 10 pairs $\{x^{(1)}, -x^{(1)}\}, \ldots, \{x^{(10)}, -x^{(10)}\}$ correctly, we can thus assume that the i-th coordinate of $x^{(i)}$ is 0. Let us choose $\{0, -1, 1\}$ as integer representatives of the residue classes mod 3. The vectors $x^{(1)}, \ldots, x^{(10)}$ can be thought of as the (reduction mod 3 of the) rows of a 10×10 integral matrix C such that

$$c_{i,i} = 0, \qquad c_{i,j} = \pm 1 \text{ for } i \neq j\,.$$

I claim that C is a *conference matrix*, i.e. $C^t \,.\, C = C \,.\, C^t = 9I$, where I is the 10×10 unit matrix.

For $i \neq j$, let $S = \{s \in \{1, \ldots, 10\} \mid x_s^{(i)} = x_s^{(j)}\}$. Clearly $i, j \notin S$. Since $w(x^{(i)} + x^{(j)}) = 2 + |S|$, and $w(x^{(i)} - x^{(j)}) = 2 + (8 - |S|)$, and the only possible values are 6 or 9, we conclude that $|S| = 4$. It follows that the scalar product of two distinct rows of C is zero.

Up to conjugation by a signed permutation matrix there is exactly one 10×10 conference matrix. Thus M_0 is uniquely determined.

It is easy to verify that there is then no choice left for the last two filling vectors (up to isomorphism of the lattices).

(3) $\mathbf{13A_2} \boxplus \mathbf{E_6}$

Here, not only is the weight polynomial determined by the duality theorem, but if we single out one of the factors $T(\mathbf{A}_2)$, the polynomial $P(x_1, x_2, y)$ corresponding to the decomposition $\mathbf{12A_2} \boxplus \mathbf{A_2} \boxplus \mathbf{E_6}$ is still uniquely determined and reads

$$P(x_1, x_2, y) = 1 + 84x_1^6 + 152x_1^9 + 6x_1^{12} + (\text{sum of monomials divisible by } x_2 \text{ or } y)\,.$$

This means that if M is an admissible metabolizer, then for any choice of coordinate (among the first 13) there must be exactly 3 pairs of vectors of weight 12 having precisely this coordinate zero.

It is then straightforward to see that we may assume these 3 pairs of vectors to be $\pm s_1$, $\pm s_2$, $\pm s_3$, where

$$\begin{aligned} s_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,0;0),\\ s_2 &= (1,1,1,1,1,1,2,2,2,2,2,2,0;0),\\ s_3 &= (1,1,1,2,2,2,1,1,1,2,2,2,0;0). \end{aligned}$$

It now turns out that the vectors with vanishing 12-th coordinate in M can then be assumed to be

$$\begin{aligned} s_4 &= (1,2,1,2,1,2,2,1,2,2,2,0,1;0)\\ s_5 &= (1,1,2,2,2,1,2,2,1,2,2,0,1;0)\\ s_1 - s_2 - s_3 + s_4 + s_5 &= (1,2,2,2,1,1,2,1,1,1,1,0,2;0) \end{aligned}$$

and their opposites, where s_1, s_2, s_3, s_4, s_5 are linearly independent and form a basis of an admissible 5-dimensional subspace in $T(13\mathbf{A}_2)$.

Indeed, among the first 11 coordinates of these 6 vectors, there must be either 4 ones and 7 twos or 4 twos and 7 ones. Since we can change the sign of the last (13-th coordinate) at will, we may assume that s_4 has the form $(1^4, 2^7, 0, 1)$, meaning 4 ones and 7 twos among the first 11 coordinates.

From the list of $\binom{11}{4} = 330$ such vectors, a sublist of 27 vectors only are compatible with s_1, s_2, s_3. Moreover, these represent a single class modulo permutations of the coordinate indices $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$ which preserve the subspace generated by s_1, s_2, s_3. Having chosen

$$s_4 = (1, 2, 1, 2, 1, 2, 2, 1, 2, 2, 2, 0, 1; 0)\ ,$$

we must select among the remaining 26 vectors compatible with s_1, s_2, s_3 together with the 27 vectors of the form $(1^4, 2^7, 0, 2; 0)$, those which are compatible with s_1, s_2, s_3, s_4. Of these, only 8 candidate vectors come out. They form a single class modulo the group generated by the permutations (1 3), (4 6), (7 9). Hence, the choice of

$$s_5 = (1, 1, 2, 2, 2, 1, 2, 2, 1, 2, 2, 0, 1; 0)$$

is also essentially unique.

Observe that $M \cap T(13\mathbf{A}_2)$ has to be 6-dimensional because the sum of the coefficients of the monomials not containing y in the weight polynomial of M is $729 = 3^6$. The search for a 6-th and last basis vector for $M \cap T(13\mathbf{A}_2)$ shows that the choice is limited to

$$s_6 = (1, 1, 2, 1, 2, 2, 2, 1, 2, 2, 0, 2, 1; 0)$$

and its 6 transforms under the group of permutations of coordinates generated by the permutations (2 3) (5 6) (8 9) and (1 2 3) (4 5 6) (7 8 9) which preserves the subspace generated by s_1, s_2, s_3, s_4, s_5.

Thus, there is essentially only one choice for $M \cap T(13\mathbf{A}_2)$. The metabolizer M itself is then easily seen to be uniquely determined.

The transformation

$$\rho(x_0, \ldots, x_{12}) = (-x_2, -x_{11}, x_7, -x_0, x_8, -x_1, x_5, x_4, -x_9, -x_{10}, x_3, x_6, x_{12})$$

carries M_0 as just described to the cyclic code of the table in Section 6.

(4)

$8\mathbf{A}_4$

Let $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$. Any metabolizer must have a basis of the form $\{e_i + v_i, i = 1, 2, 3, 4\}$ for some vectors $v_i \in \mathbf{F}_5^4$ of weight 3 or 4.

Hence, we may assume that the first basis vector is either $s_1 = e_1 + (1, 1, 1, 1)$ or $t_1 = e_1 + (0, 1, 2, 2)$.

If we start with s_1, there are essentially only 2 ways of completing s_1 to an admissible metabolizer with 3 vectors forming with s_1 the rows of the matrix S exhibited in the table and the matrix S':

$$S' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 \end{pmatrix}.$$

If we start with t_1 there is essentially only one way to complete to a metabolizer:

$$S'' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 & 0 \end{pmatrix}.$$

All these metabolizers are equivalent. The permutation ρ defined by

$$\rho(x_0, \ldots, x_7) = (x_4, x_1, x_2, -x_3, x_7, x_5, x_6, x_0)$$

sends S' to S and σ defined by

$$\sigma(x_0, \ldots, x_7) = (x_5, x_1, x_4, x_0, x_7, x_2, x_3, x_6)$$

sends S'' to S.

Thus the lattice described by the filling set S is the only one with the root system $8\mathbf{A}_4$.

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