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Artikel: PROOF OF MARGULIS' THEOREM ON VALUES OF QUADRATIC FORMS, INDEPENDENT OF THE AXIOM OF CHOICE

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in $C(X)$ this readily implies that $n_k^{-1} \sum_{i=0}^{n_k - 1} f \circ \varphi^i(x_0)$ converges for all $f \in C(X)$; let c_f be the limit corresponding to f. Then it can be verified that $\Lambda: C(X) \to \mathbf{R}$ defined by $\Lambda(f) = c_f$, for all $f \in C(X)$, is a φ-invariant integral on $C(X)$. Also clearly Λ is not identically zero and therefore by our observations above, the support, say X', is a nonempty closed φ-invariant subset of X and further $C(X')$ admits an integral with full support (namely X') which is invariant under the restriction of φ to X'. Replacing X as in the hypothesis by X' we may without loss of generality assume that $C(X)$ admits a φ-invariant integral whose support is X; in the rest of the argument we let Λ be any such integral.

Now suppose that there do not exist any recurrent points for φ. Let $\rho(\cdot, \cdot)$ be the metric on X. Let θ be the function on X defined by $\theta(x) = \inf\{\rho(\varphi^i(x), x) \mid i = 1, 2, \ldots\}$, for all $x \in X$. There being no recurrent points means that $\theta(x) > 0$ for all $x \in X$. For each natural number k let $E_k = \{x \in X \mid \theta(x) \geqslant 1/k\}$. Then each E_k is a closed subset of X and $X = \cup E_k$. Therefore by the Baire category theorem there exists a k such that E_k has an interior point in X. In particular, there exists an open ball, say A, of radius at most $1/3k$ contained in E_k. The definition of E_k and the condition on the radius of A then imply that the sets $\varphi^i(A)$, $i \in \mathbf{Z}$, are mutually disjoint. Now let $x \in A$ and let $f \in C(X)^+$ be such that $f(x) > 0$ and the support of f (the closure of the set $\{y \in X \mid f(y) > 0\}$) is contained in A. For each natural number n let $S_n(f) = \sum_{i=0}^{n-1} f \circ \varphi^i \in C(X)$. The disjointness of $\varphi^i(A)$, $i \in \mathbf{Z}$, implies that, for any n, $\| S_n(f) \| = \| f \|$. Also, by the φ-invariance of Λ we have $\Lambda(S_n(f)) = n\Lambda(f)$. Hence $\Lambda(f) = \Lambda(S_n(f))/n \leqslant \| S_n(f) \| \Lambda(1_X)/n = \| f \| \Lambda(1_X)/n$ for all n, where 1_X denotes the constant function with value 1. But this implies that $\Lambda(f) = 0$ contradicting the assumption that the support of Λ is the whole of X. This proves the proposition.

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