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*Proof.* Note that  $2^n - 2^m \geq 2^m$  if  $n > m$  and that  $2^{n_1} - 2^{m_1} = 2^{n_2} - 2^{m_2}$  if and only if  $(n_1, m_1) = (n_2, m_2)$ . So whenever  $n = \frac{k(k-1)}{2}$  for some  $k$  we have

$$\left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| = \left\| \prod_{1 \leq i < j \leq k} (z^{2j-1} - z^{2i-1}) \right\| \leq k^{k/2} \leq \sqrt{2n}^{\sqrt{n/2}}.$$

While if  $\frac{k(k-1)}{2} < n < \frac{(k+1)k}{2}$  then

$$\begin{aligned} \left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| &\leq \left\| \prod_{1 \leq i < j \leq k} (z^{2j-1} - z^{2i-1}) \right\| \left\| \prod_{i=\frac{k(k-1)}{2}+1}^n (1 - z^{\beta_i}) \right\| \\ &\leq \sqrt{2n}^{\sqrt{n/2}} 2^{n - \frac{k(k-1)}{2} - 1} \leq \sqrt{2n}^{\sqrt{n/2}} 2^{k-1} \\ &\leq \sqrt{2n}^{\sqrt{n/2}} 2^{\sqrt{2n}} = (32n)^{\sqrt{n/8}}. \end{aligned} \quad \square$$

This is not as good an estimate as Odlyzko's in [16] (see also [13]) which has exponent roughly  $n^{1/3}$ . What distinguishes it is that it holds for all the partial products of a single infinite product (with distinct increasing exponents). Also, clearly any  $\alpha > 2$  could play the role of 2 in the construction of the  $\beta_i$  with the exact same conclusion.

**THEOREM 1.** *Let  $\{\delta_i\}$  be any sequence of integers and let  $\{\beta_i\}$  be the sequence of differences in the following order*

$$\{\delta_1 - \delta_0, \delta_2 - \delta_0, \delta_2 - \delta_1, \dots, \delta_n - \delta_0, \dots, \delta_n - \delta_{n-1}, \dots\}$$

*then*

$$\left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| \leq (32n)^{\sqrt{n/8}}.$$

## 5. PERFECT SOLUTIONS OF PRIME SIZE

The first unresolved case of the Prouhet-Tarry-Escott problem is the eleven case. The previous ideal solutions were all found without computer assistance; indeed the cases 1, ..., 10 were all resolved prior to 1950. It therefore seems appropriate to discuss an algorithm for searching for such solutions. We wish to perform a computer search for perfect symmetric ideal solutions

of size 11. To this end we produce a method of finding all such solutions mod  $11^n$  for any  $n$ . As this method applies to any odd prime  $p$  we present it in the general situation. (A similar method for solving the ideal Prouhet-Tarry-Escott problem mod  $p^n$  is suggested in [17] for all primes  $p$  greater or equal to the size.) We will be using symmetric residues throughout, as they facilitate checking for solutions in ranges of the form  $[-l, l]$ .

LEMMA 7. *If  $\{\beta_0, \dots, \beta_{p-1}\}$  is a perfect solution mod  $p^{n+1}$  then*

$$\beta_i = m_i p^n + \alpha_i \quad \text{for } i = 0, \dots, p-1$$

*and  $\{\alpha_0, \dots, \alpha_{p-1}\}$  is a perfect solution mod  $p^n$ .*

*Proof.* This is done by expanding  $\{\beta_0, \dots, \beta_{p-1}\}$  to the base  $p$ . □

This simple lemma allows us to create solutions mod  $p^n$  for any  $n$  inductively. We only need to find the  $\{m_0, \dots, m_{p-1}\}$  given  $\{\alpha_0, \dots, \alpha_{p-1}\}$ . This is provided by the theorem below.

Now suppose that  $\{\alpha_0, \dots, \alpha_{p-1}\}$  is a perfect solution mod  $p^n$ . We define

$$s_k = - \frac{\sum_{i=0}^{p-1} \alpha_i^{2k-1}}{p^n} \quad \text{for } k = 1, \dots, \frac{p-1}{2}.$$

We also suppose without loss of generality that  $\alpha_i \equiv i \pmod{p}$  for  $i = 0, \dots, p-1$ .

THEOREM 2. *Given  $\{\alpha_0, \dots, \alpha_{p-1}\}$ , a perfect solution mod  $p^n$ , all  $p^{\frac{p+1}{2}}$  perfect solutions mod  $p^{n+1}$  of the form*

$$\{m_0 p^n + \alpha_0, \dots, m_{p-1} p^n + \alpha_{p-1}\}$$

*are given by*

$$(m_0, \dots, m_{p-1}) = (a_0, \dots, a_{p-1}) + (h_0, \dots, h_{p-1}),$$

*where*

$$a_0 = 0$$

$$a_i = \sum_{j=1}^{p-1} \frac{i^{2-2j}}{2j-1} s_j \pmod{p} \quad \text{for } i = 1, \dots, \frac{p-1}{2}$$

$$a_i = a_{p-i} \quad \text{for } i = \frac{p+1}{2}, \dots, p-1$$

and  $(h_0, \dots, h_{\frac{p-1}{2}})$  are arbitrary residues mod  $p$  and

$$h_i = 2h_0 - h_{p-i} \quad \text{for } i = \frac{p+1}{2}, \dots, p-1.$$

So there are exactly  $p^{\frac{p+1}{2}}$  perfect solutions mod  $p^{n+1}$ .

*Proof.* Suppose  $\{m_i p^n + \alpha_i\}$  is a perfect solution mod  $p^{n+1}$  and  $\{\alpha_i\}$  is a perfect solution mod  $p^n$ . For  $k = 1, \dots, \frac{p-1}{2}$

$$\sum_{i=0}^{p-1} (m_i p^n + \alpha_i)^{2k-1} \equiv 0 \pmod{p^{n+1}}.$$

On expanding we get

$$\begin{aligned} \sum_{i=0}^{p-1} ((2k-1)\alpha_i^{2k-2} m_i p^n + \alpha_i^{2k-1}) &\equiv 0 \pmod{p^{n+1}} \\ \sum_{i=0}^{p-1} (2k-1)\alpha_i^{2k-2} m_i p^n &\equiv - \sum_{i=0}^{p-1} \alpha_i^{2k-1} \pmod{p^{n+1}}. \end{aligned}$$

Division by  $p^n$  gives us

$$\sum_{i=0}^{p-1} (2k-1)\alpha_i^{2k-2} m_i \equiv - \frac{\sum_{i=0}^{p-1} \alpha_i^{2k-1}}{p^n} \pmod{p},$$

and since  $\alpha_i \equiv i \pmod{p}$  we have

$$\sum_{i=0}^{p-1} (2k-1)i^{2k-2} m_i \equiv - \frac{\sum_{i=0}^{p-1} i^{2k-1}}{p^n} \pmod{p}.$$

So we define  $A$ , a  $(\frac{p-1}{2} \times p)$  matrix, by

$$A_{k,i} \equiv (2k-1)(i-1)^{2k-2} \pmod{p}.$$

We now have, with  $s := (s_0, \dots, s_{(p-1)/2})$  and  $m := (m_0, \dots, m_{(p-1)})$ ,

$$Am \equiv s \pmod{p}.$$

For example with  $p = 7$  we get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & -2 & -1 & -1 & -2 & 3 \\ 0 & -2 & 3 & -1 & -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_6 \end{pmatrix} = \begin{pmatrix} \sum \alpha_i \\ \sum \alpha_i^3 \\ \sum \alpha_i^5 \end{pmatrix}.$$

In general the rank of  $A$  is always  $\frac{p-1}{2}$ , as the next argument makes clear, so there are  $p^{\frac{p+1}{2}}$  solutions of this underdetermined linear system.

We first derive a particular solution  $a := (a_0, \dots, a_{p-1})$  of the system. We set  $a_0 = 0$  and  $\bar{A}$  to be  $A$  without its first column. We also define  $\bar{a}$  to be  $a$  without  $a_0$ . We solve the reduced system

$$\bar{A}\bar{a} \equiv s \pmod{p}$$

by the standard method. So

$$\bar{a} \equiv \bar{A}^T(\bar{A}\bar{A}^T)^{-1}s \pmod{p}.$$

$\bar{A}\bar{A}^T$  is a particularly simple symmetric matrix given by

$$\begin{pmatrix} \sum i^0 & \sum 3i^2 & \sum 5i^4 & \dots & \sum (p-2)i^{p-3} \\ \vdots & \sum 9i^4 & \sum 15i^6 & \dots & \sum 3(p-2)i^{p-1} \\ \vdots & \vdots & \sum 25i^8 & \dots & \sum 5(p-2)i^{p+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \sum (p-2)^2 i^{2p-6} \end{pmatrix}$$

where each sum ranges over  $i = 1, \dots, p-1$ . Since  $\sum_{i=1}^{p-1} i^k \equiv 0 \pmod{p}$  when  $k \not\equiv 0 \pmod{p-1}$  almost all the elements of the matrix vanish and we are left with a very simple matrix. In fact we get the product of a diagonal and a permutation matrix. Note that this shows that  $A$  has full rank modulo  $p$ . For example when  $p = 11$  we get

$$\bar{A}\bar{A}^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \end{pmatrix}.$$

So it is a simple matter to find  $B = \bar{A}^T(\bar{A}\bar{A}^T)^{-1}$ . For  $i = 1, \dots, p-1$   $j = 1, \dots, \frac{p-1}{2}$

$$B_{i,j} \equiv \frac{-i^{2-2j}}{2j-1} \pmod{p}.$$

For example  $B$ , when  $p = 7$ , is

$$\begin{pmatrix} -1 & 2 & -3 \\ -1 & -3 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -3 & 2 \\ -1 & 2 & -3 \end{pmatrix}.$$

So our particular solution  $a$  is given by  $a_0 = 0$  and  $\bar{a} = Bs$ .

To find the solution  $h$  of the homogeneous system

$$Ah \equiv 0 \pmod{p}$$

consider the reduced system

$$\bar{A}\bar{h} \equiv \begin{pmatrix} -h_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{p}.$$

Note that if  $h_i + h_{p-i} \equiv 2h_0$  for  $i = 1, \dots, \frac{p-1}{2}$  we have a solution since

$$\sum_{i=1}^{p-1} i^k \equiv 0 \pmod{p} \quad \text{if} \quad k \not\equiv 0 \pmod{p-1}.$$

Finally setting  $(h_0, h_1, \dots, h_{\frac{p-1}{2}})$  arbitrary we get the solution as in the statement of the theorem.  $\square$

This theorem allows one to calculate all  $p^{(n-1)\frac{p+1}{2}}$  perfect solutions mod  $p^n$  for any odd prime  $p$  and any  $n$ . This is essentially calculating solutions in the ring of  $p$ -adic integers. We were hoping to find a perfect solution of size 11 using this method, but we were only able to show that there is no such solution with coefficients in the range  $[-363, 363]$ . This is because there are  $11^6$  solutions mod  $11^2$ , and  $11^{12}$  solutions mod  $11^3$ . So checking for solutions in the relatively small range  $[-665, 665]$ , would require checking more than a billion cases. Even checking in the range  $[-363, 363]$  was a substantial computation. We were able to compute all  $7^8$  solutions mod  $7^3$  to find that all perfect solutions of size 7 with coefficients in the range  $[-171, 171]$  are

$$\begin{aligned} &\{-51, -33, -24, 7, 13, 38, 50\} \\ &\{-90, -86, -39, -5, 48, 77, 95\} \\ &\{-116, -104, -36, -19, 75, 77, 123\} \\ &\{-120, -110, -23, -13, 38, 105, 123\} \\ &\{-134, -75, -66, 8, 47, 87, 133\}. \end{aligned}$$

We hope that this technique in combination with others may yield a viable computer search for a perfect solution of size 11.