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PROPOSITION 8. *For symmetric solutions we have*

$$19 \mid r_7, \quad 19 \mid r_{11}, \quad 17 \cdot 19 \mid r_{13}$$

Proof. This is a result of performing the calculation mod p and observing that $C_n \equiv 0 \pmod{p}$. \square

It is interesting to observe that an ideal solution in its third form has a large factor

$$\prod (1 - x^{p_i}).$$

This follows from Propositions 6 and 7. Hence the degree of this polynomial grows at least like $n^2/(2 \log n)$.

4. RELATED PROBLEMS

There are several related problems. We mention two.

4.1. THE ‘EASIER’ WARING PROBLEM

In [21] Wright stated, and probably misnamed, the following variation of the well known Waring problem. The problem is to find the least s so that for all n there are natural numbers $\{\alpha_1, \dots, \alpha_s\}$ so that

$$\pm \alpha_1^k \pm \cdots \pm \alpha_s^k = n$$

for some choice of signs. We denote the least such s by $v(k)$. Recall that the usual Waring problem requires all positive signs. For arbitrary k the best known bounds for $v(k)$ derive from the bounds for the usual Waring problem. So to date, the “easier” Waring problem is not easier than the Waring problem. However, the best bounds for small k are derived in an elementary manner from solutions to the Prouhet-Tarry-Escott problem.

Suppose $\{\alpha_1, \dots, \alpha_n\} \stackrel{k-2}{=} \{\beta_1, \dots, \beta_n\}$. We see that

$$\sum_{i=1}^n (x + \alpha_i)^k - \sum_{i=1}^n (x + \beta_i)^k = Cx + D$$

where

$$C = k \left(\sum_{i=1}^n \alpha_i^{k-1} - \sum_{i=1}^n \beta_i^{k-1} \right)$$

and

$$D = \sum_{i=1}^n \alpha_i^k - \sum_{i=1}^n \beta_i^k.$$

We define $\Delta(k, C)$ to be the smallest s such that every residue mod C is represented by s positive and negative k^{th} powers. We also define $\Delta(k) = \max_C \Delta(k, C)$. Wright shows how to calculate $\Delta(k, C)$ and $\Delta(k)$ in [9].

LEMMA 4. *If*

$$\sum_{i=1}^n (x + \alpha_i)^k - \sum_{i=1}^n (x + \beta_i)^k = Cx + D$$

then

$$v(k) \leq 2n + \Delta(k, C) \leq 2n + \Delta(k).$$

Proof. This follows directly from the above definitions. \square

PROPOSITION 9.

$$\begin{aligned} v(k) &\leq 2M(k-2) + \Delta(k) \leq 2(k-1) \left(\frac{\log^{\frac{1}{2}}(k)}{\log\left(1 + \frac{1}{k-2}\right)} + 1 \right) \\ &\quad + \begin{cases} \frac{1}{2}(3k-1) & k \text{ odd} \\ 2k & k \text{ even} \end{cases} \end{aligned}$$

Proof. This follows from the fact that

$$\Delta(k) \leq \begin{cases} \frac{1}{2}(3k-1) & k \text{ odd} \\ 2k & k \text{ even} \end{cases}$$

which is established in [22], and Lemma 4, and Hua's bound for $M(k)$ in [11]. Note that we must use $M(k)$ and not $N(k)$ since we require exact solutions so that $C \neq 0$. \square

The best bounds for small k are derived from the above lemma using specific solutions of the Prouhet-Tarry-Escott problem and careful computation of $\Delta(k, C)$. In the following table we represent solutions as in the third form of the problem, and we define

$$[n_1, \dots, n_k] := \prod_{i=1}^k (1 - x^{n_i})$$

$$g := 1 - x + x^3 + x^5 - x^4 + x^{10} + x^{27} + x^{17} - x^{26} - x^{23} + x^{22} + x^{24}$$

$$h := x + x^{25} + x^{31} + x^{84} + x^{87} + x^{134} + x^{158} + x^{182} + x^{198}$$

$$- x^2 - x^{18} - x^{42} - x^{66} - x^{113} - x^{116} - x^{169} - x^{175} - x^{199}$$

k bound for $v(k)$ solution

7	14	[1, 1, 2, 3, 4, 5]
8	30	[3, 5, 7, 11, 13, 17, 19] $\cdot g$
9	29	[1, 2, 3, 5, 7, 8, 11, 13]
10	30	h
11	28	[1, 2, 3, 4, 5, 7, 9, 11, 13, 17]
12	37	[1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19]
13	39	[1, 2, 3, 5, 6, 7, 8, 9, 11, 13, 17, 19]
14	53	[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 17, 19]
15	69	[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19]
16	92	[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19]
17	72	[1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 19]
18	86	[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29]
19	88	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23]
20	120	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 29]

This table is from [9] and [24] as are most of the results of this section. Some of the bounds are improved by using Wright's calculation of $\Delta(k)$ and our solutions of smaller size.

4.2. A PROBLEM OF ERDŐS AND SZEKERES

We call a solution $\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}$ of the Prouhet-Tarry-Escott problem a **pure product** if

$$\sum_{i=1}^n z^{\alpha_i} - \sum_{i=1}^n z^{\beta_i} = \prod_{i=1}^k (1 - z^{n_i})$$

for some n_1, \dots, n_k . Note that pure products are obtained from ideal solutions of degree zero by applying Lemma 2 repeatedly. These are a very restricted class of solutions of the Prouhet-Tarry-Escott Problem.

PROPOSITION 10. *If*

$$\sum_{i=1}^n z^{\alpha_i} - \sum_{i=1}^n z^{\beta_i} = \prod_{i=1}^k (1 - z^{n_i})$$

then $\{\alpha_i\}, \{\beta_i\}$ *is equivalent to a symmetric solution of degree k and size n .*

Proof. Note that symmetry in the third form of the problem requires

$$f(z) = \sum_{i=1}^n z^{\alpha_i} - \sum_{i=1}^n z^{\beta_i} = (-1)^k f(1/z).$$

The appropriate equivalent solution can be shown to satisfy this condition. \square

For $f(z) = \prod_{i=1}^k (1 - z^{n_i}) = \sum_{i=0}^n \alpha_i z^i$, where $n = \deg f$, we define the norms

$$\begin{aligned} \|f\|_1 &= \sum_{i=0}^n |\alpha_i| \\ \|f\|_2 &= \left(\sum_{i=0}^n \alpha_i^2 \right)^{1/2} = \left(\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})^2 d\theta \right)^{1/2} \\ \|f\|_\infty &= \sup_{|z|=1} |f(z)|. \end{aligned}$$

We observe that $\|f\|_1$ is twice the size of the solution $\{\alpha_i\}, \{\beta_i\}$ of the Prouhet-Tarry-Escott problem.

LEMMA 5.

$$\frac{\|f\|_1}{\sqrt{\deg f + 1}} \leq \|f\|_2 \leq \|f\|_\infty \leq \|f\|_1 \leq \|f\|_2^2.$$

Proof. This is all easily established. It all follows from well known inequalities and the fact that the coefficients of f are integers. \square

In 1958 [8] Erdős and Szekeres formulated the problem of finding

$$A(k) = \min_{n_1, \dots, n_k} \left\| \prod_{i=1}^k (1 - z^{n_i}) \right\|_\infty$$

They have conjectured that $A(k) \geq k^C$ for any C . There has been very little progress in this pretty old problem. Though an interesting and possibly related problem is solved in [2]. See Section 6.

We can use pure product solutions of the Prouhet-Tarry-Escott problem to find upper bounds for $A(k)$. These are not good general bounds, but we do find good upper bounds for small values of k using specific solutions. The following table was derived using various greedy algorithms to find the $\{n_i\}$.

k	$\ f\ _1$	$\{n_1, \dots, n_k\}$
1	2	{1}
2	4	{1, 2}
3	6	{1, 2, 3}
4	8	{1, 2, 3, 4}
5	10	{1, 2, 3, 5, 7}
6	12	{1, 1, 2, 3, 4, 5}
7	16	{1, 2, 3, 4, 5, 7, 11}
8	16	{1, 2, 3, 5, 7, 8, 11, 13}
9	20	{1, 2, 3, 4, 5, 7, 9, 11, 13}
10	24	{1, 2, 3, 4, 5, 7, 9, 11, 13, 17}
11	28	{1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19}
12	36	{1, ..., 9, 11, 13, 17}
13	48	{1, ..., 9, 11, 13, 17, 19}
14	56	{1, ..., 7, 9, 10, 11, 13, 15, 16, 17}
15	60	{1, ..., 7, 9, 10, 11, 13, 15, 16, 17, 19}
16	60	{1, ..., 11, 13, 15, 17, 19, 23}
17	68	{1, ..., 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29}
18	84	{1, ..., 11, 13, 14, 16, 17, 19, 22, 23}
19	100	{1, ..., 11, 13, 15, 17, 19, 21, 23, 25, 29}
20	116	{1, ..., 11, 13, 15, 17, 19, 21, 23, 25, 27, 31}
21	130	{1, ..., 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31}
22	140	{1, ..., 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37}
23	156	{1, ..., 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37}
24	204	{1, ..., 7, 9, 10, 11, 13, 15, 16, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37}
25	188	{1, ..., 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 41}
26	228	{1, ..., 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41}
27	276	{1, ..., 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41}
28	336	{1, ..., 13, 15, 17, 18, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41}
29	392	{1, 1, 2, 2, ..., 27}
30	432	{1, 1, 1, 2, ..., 28}

k	$\ f\ _1$	$\{n_1, \dots, n_k\}$
40	1900	$\{1, 2, 2, \dots, 17, 19, \dots, 29, 31, \dots, 37, 43, 47, 49, 49\}$
41	1348	$\{1, 2, 2, \dots, 17, 19, \dots, 29, 31, \dots, 38, 40, 43, 49, 53\}$
42	1936	$\{1, 2, 2, \dots, 17, 19, \dots, 29, 31, \dots, 38, 40, 43, 47, 52, 53\}$
43	2396	$\{1, 2, 2, \dots, 17, 19, \dots, 29, 31, \dots, 38, 40, 43, 46, 52, 53, 60\}$
44	2492	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 43, 46, 52, 53, 60\}$
45	2684	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 43, 44, 46, 52, 53, 60\}$
46	2336	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 43, 44, 46, 48, 52, 53, 60\}$
47	3196	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 40, 43, 44, 46, 48, 52, 53, 60\}$
48	4080	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 40, 43, 44, 46, 48, 50, 52, 53, 60\}$
49	4086	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 40, 43, 44, 46, 48, 50, 52, 53, 55, 60\}$
50	5088	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 40, 43, 44, 46, 48, 49, 50, 52, 53, 55, 60\}$
51	5480	$\{1, 2, 2, \dots, 29, 31, \dots, 38, 40, 40, 43, 44, 46, 48, 49, 50, 52, 53, 55, 56, 60\}$
52	5296	$\{1, \dots, 11, 13, 16, 17, 24, 52, \dots, 56, \dots, 58, 80, 82, 83, 84, 86, 88, 89, 92, 95, 100\}$
53	6000	$\{1, \dots, 11, 13, 16, 17, 24, 52, 53, 54, 56, 58, \dots, 80, 82, 83, 84, 86, 88, 89, 90, 92, 95, 100, 142\}$
54	7352	$\{1, 1, 2, 2, \dots, 29, 31, \dots, 38, 40, 42, 43, 44, 46, 48, \dots, 53, 55, 56, 60\}$
55	5044	$\{1, 1, 2, 2, \dots, 29, 31, \dots, 38, 40, 42, 43, 44, 46, \dots, 56, 60\}$
56	7536	$\{1, 1, \dots, 11, 13, 16, 17, 24, 52, 53, 54, 56, 58, \dots, 80, 82, \dots, 92, 95, 100\}$
57	7156	$\{1, 1, \dots, 11, 13, 16, 17, 24, 52, \dots, 56, 58, \dots, 80, 82, \dots, 92, 95, 100\}$
58	6268	$\{1, 1, 2, 2, \dots, 29, 31, \dots, 38, 41, \dots, 44, 46, \dots, 60\}$
59	7572	$\{1, 1, \dots, 11, 13, \dots, 17, 24, 52, \dots, 52, 58, \dots, 80, 82, \dots, 92, 95, 100\}$
60	10848	$\{1, 1, \dots, 11, 13, \dots, 17, 24, 52, \dots, 56, 58, \dots, 80, 82, \dots, 92, 95, 100, 100\}$
80	1629900	$\{1, \dots, 73, 90, \dots, 95, 97\}$
100	41947220	$\{1, \dots, 89, 107, \dots, 117\}$

For $k = 1, 2, 3, 4, 5, 6$, and 8 these products are ideal solutions and therefore also optimal. These may well be the only k for which pure products give ideal solutions. We computed extensively on degree 6 ($k = 7$) and could not find a degree 6 product with $\|f\|_1 = 14$. Since $\|f\|_1$ is always an even integer we therefore conjecture that the minimum attainable is 16 (as above). For larger k there is no reason to believe that we have found minimal examples. This table also provides some good bounds for $N(k)$. For example $N(29) \leq 216$ which is much better than the bound of 419 that derives from the discussion following Proposition 3. There are many partial results on the Erdős-Szemerédi problem

to be found in [8], [1], [6], [14], [3], [20], [2], [16] and [13]. We give one such new result here.

We now construct an easy example to show that we cannot in general expect exponential growth of the norms of the partial products of $\prod_{i=1}^{\infty} (1 - z^{\beta_i})$ on the unit disk. From this point on, $\|f\|$ without a subscript will denote $\|f\|_{\infty}$.

LEMMA 6. *Let $1 \leq \beta_1 < \beta_2 < \dots$ and let*

$$W_n(z) = \prod_{1 \leq i < j \leq n} (1 - z^{\beta_j - \beta_i})$$

then

$$\|W_n(z)\| \leq n^{\frac{n}{2}}.$$

Proof. We can explicitly evaluate the Vandermonde determinant

$$D_n := \prod_{1 \leq i < j \leq n} (z^{\beta_j} - z^{\beta_i}) = \begin{vmatrix} 1 & z^{\beta_1} & \cdots & z^{(n-1)\beta_1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & z^{\beta_n} & \cdots & z^{(n-1)\beta_n} \end{vmatrix}$$

and by Hadamard's inequality, since each entry of the matrix has modulus at most one in the unit disk,

$$\|D_n\| \leq n^{n/2}.$$

Thus

$$\left\| \prod_{1 \leq i < j \leq n} (1 - z^{\beta_j - \beta_i}) \right\| = \left\| \prod_{1 \leq i < j \leq n} (z^{\beta_j} - z^{\beta_i}) \right\| \leq n^{n/2}. \quad \square$$

Observe, as Dobrowolski did in [6], that if we take $\beta_i = i$, we deduce that

$$\left\| \prod_{i=1}^n (1 - z^i)^{n-i-1} \right\| \leq n^{n/2},$$

a result originally obtained by Atkinson in [1].

PROPOSITION 11. *Let β_i be the sequence formed by taking the set $\{2^n - 2^m : n > m \geq 0\}$ in increasing order. Then for all n ,*

$$\left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| \leq (32n)^{\sqrt{n/8}}.$$

Proof. Note that $2^n - 2^m \geq 2^m$ if $n > m$ and that $2^{n_1} - 2^{m_1} = 2^{n_2} - 2^{m_2}$ if and only if $(n_1, m_1) = (n_2, m_2)$. So whenever $n = \frac{k(k-1)}{2}$ for some k we have

$$\left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| = \left\| \prod_{1 \leq i < j \leq k} (z^{2^{j-1}} - z^{2^{i-1}}) \right\| \leq k^{k/2} \leq \sqrt{2n}^{\sqrt{n/2}}.$$

While if $\frac{k(k-1)}{2} < n < \frac{(k+1)k}{2}$ then

$$\begin{aligned} \left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| &\leq \left\| \prod_{1 \leq i < j \leq k} (z^{2^{j-1}} - z^{2^{i-1}}) \right\| \left\| \prod_{i=\frac{k(k-1)}{2}+1}^n (1 - z^{\beta_i}) \right\| \\ &\leq \sqrt{2n}^{\sqrt{n/2}} 2^{n - \frac{k(k-1)}{2} - 1} \leq \sqrt{2n}^{\sqrt{n/2}} 2^{k-1} \\ &\leq \sqrt{2n}^{\sqrt{n/2}} 2^{\sqrt{2n}} = (32n)^{\sqrt{n/8}}. \end{aligned} \quad \square$$

This is not as good an estimate as Odlyzko's in [16] (see also [13]) which has exponent roughly $n^{1/3}$. What distinguishes it is that it holds for all the partial products of a single infinite product (with distinct increasing exponents). Also, clearly any $\alpha > 2$ could play the role of 2 in the construction of the β_i with the exact same conclusion.

THEOREM 1. *Let $\{\delta_i\}$ be any sequence of integers and let $\{\beta_i\}$ be the sequence of differences in the following order*

$$\{\delta_1 - \delta_0, \delta_2 - \delta_0, \delta_2 - \delta_1, \dots, \delta_n - \delta_0, \dots, \delta_n - \delta_{n-1}, \dots\}$$

then

$$\left\| \prod_{i=1}^n (1 - z^{\beta_i}) \right\| \leq (32n)^{\sqrt{n/8}}.$$

5. PERFECT SOLUTIONS OF PRIME SIZE

The first unresolved case of the Prouhet-Tarry-Escott problem is the eleven case. The previous ideal solutions were all found without computer assistance; indeed the cases 1, ..., 10 were all resolved prior to 1950. It therefore seems appropriate to discuss an algorithm for searching for such solutions. We wish to perform a computer search for perfect symmetric ideal solutions