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Artikel: THE PROUHET-TARRY-ESCOTT PROBLEM REVISITED

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PROPOSITION 8. *For symmetric solutions we have*

$$19 \mid r_7, \quad 19 \mid r_{11}, \quad 17 \cdot 19 \mid r_{13}$$

Proof. This is a result of performing the calculation mod p and observing that $C_n \equiv 0 \bmod p$. □

It is interesting to observe that an ideal solution in its third form has a large factor

$$\prod (1 - x^{p_i}) .$$

This follows from Propositions 6 and 7. Hence the degree of this polynomial grows at least like $n^2/(2 \log n)$.

4. RELATED PROBLEMS

There are several related problems. We mention two.

4.1. THE 'EASIER' WARING PROBLEM

In [21] Wright stated, and probably misnamed, the following variation of the well known Waring problem. The problem is to find the least s so that for all n there are natural numbers $\{\alpha_1, \ldots, \alpha_s\}$ so that

$$\pm \alpha_1^k \pm \cdots \pm \alpha_s^k = n$$

for some choice of signs. We denote the least such s by $v(k)$. Recall that the usual Waring problem requires al positive signs. For arbitrary k the best known bounds for $v(k)$ derive from the bounds for the usual Waring problem. So to date, the "easier" Waring problem is not easier than the Waring problem. However, the best bounds for small k are derived in an elementary manner from solutions to the Prouhet-Tarry-Escott problem.

Suppose $\{\alpha_1, \ldots, \alpha_n\} \stackrel{k-2}{=} \{\beta_1, \ldots, \beta_n\}$. We see that

$$\sum_{i=1}^{n} (x + \alpha_i)^k - \sum_{i=1}^{n} (x + \beta_i)^k = Cx + D$$

where

$$C = k \left(\sum_{i=1}^{n} \alpha_i^{k-1} - \sum_{i=1}^{n} \beta_i^{k-1} \right)$$

and

$$D = \sum_{i=1}^{n} \alpha_i^k - \sum_{i=1}^{n} \beta_i^k .$$

We define $\Delta(k, C)$ to be the smallest s such that every residue mod C is represented by s positive and negative k^{th} powers. We also define $\Delta(k) = \max_C \Delta(k, C)$. Wright shows how to calculate $\Delta(k, C)$ and $\Delta(k)$ in [9].

LEMMA 4. *If*

$$\sum_{i=1}^{n} (x + \alpha_i)^k - \sum_{i=1}^{n} (x + \beta_i)^k = Cx + D$$

then

$$v(k) \leqslant 2n + \Delta(k, C) \leqslant 2n + \Delta(k) .$$

Proof. This follows directly from the above definitions. □

PROPOSITION 9.

$$v(k) \leqslant 2M(k-2) + \Delta(k) \leqslant 2(k-1)\left(\frac{\log \frac{1}{2}(k)}{\log\left(1 + \frac{1}{k-2}\right)} + 1\right) + \begin{cases} \frac{1}{2}(3k-1) & k \text{ odd} \\ 2k & k \text{ even} . \end{cases}$$

Proof. This follows from the fact that

$$\Delta(k) \leqslant \begin{cases} \frac{1}{2}(3k-1) & k \text{ odd} \\ 2k & k \text{ even} \end{cases}$$

which is established in [22], and Lemma 4, and Hua's bound for $M(k)$ in [11]. Note that we must use $M(k)$ and not $N(k)$ since we require exact solutions so that $C \neq 0$. □

The best bounds for small k are derived from the above lemma using specific solutions of the Prouhet-Tarry-Escott problem and careful computation of $\Delta(k, C)$. In the following table we represent solutions as in the third form of the problem, and we define

$$[n_1, \ldots, n_k] := \prod_{i=1}^{k} (1 - x^{n_i})$$

$$g := 1 - x + x^3 + x^5 - x^4 + x^{10} + x^{27} + x^{17} - x^{26} - x^{23} + x^{22} + x^{24}$$

$$h := x + x^{25} + x^{31} + x^{84} + x^{87} + x^{134} + x^{158} + x^{182} + x^{198} - x^2 - x^{18} - x^{42} - x^{66} - x^{113} - x^{116} - x^{169} - x^{175} - x^{199}$$

k	bound for $\upsilon(k)$	solution
7	14	[1, 1, 2, 3, 4, 5]
8	30	[3, 5, 7, 11, 13, 17, 19] $\cdot g$
9	29	[1, 2, 3, 5, 7, 8, 11, 13]
10	30	h
11	28	[1, 2, 3, 4, 5, 7, 9, 11, 13, 17]
12	37	[1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19]
13	39	[1, 2, 3, 5, 6, 7, 8, 9, 11, 13, 17, 19]
14	53	[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 17, 19]
15	69	[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19]
16	92	[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19]
17	72	[1, 1, 2, 3, 4, 5, 6, 7, 7, 8, 9, 10, 11, 13, 17, 19]
18	86	[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29]
19	88	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23]
20	120	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 29]

This table is from [9] and [24] as are most of the results of this section. Some of the bounds are improved by using Wright's calculation of $\Delta(k)$ and our solutions of smaller size.

4.2. A Problem of Erdős and Szekeres

We call a solution $\{\alpha_1, \ldots, \alpha_n\}$, $\{\beta_1, \ldots, \beta_n\}$ of the Prouhet-Tarry-Escott problem a **pure product** if

$$\sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{\beta_i} = \prod_{i=1}^{k} (1 - z^{n_i})$$

for some $n_1, \ldots, n_k$. Note that pure products are obtained from ideal solutions of degree zero by applying Lemma 2 repeatedly. These are a very restricted class of solutions of the Prouhet-Tarry-Escott Problem.

Proposition 10. *If*

$$\sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{\beta_i} = \prod_{i=1}^{k} (1 - z^{n_i})$$

then $\{\alpha_i\}$, $\{\beta_i\}$ *is equivalent to a symmetric solution of degree* k *and size* n.

Proof. Note that symmetry in the third form of the problem requires

$$f(z) = \sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{\beta_i} = (-1)^k f(1/z) .$$

The appropriate equivalent solution can be shown to satisfy this condition. □

For $f(z) = \prod_{i=1}^{k}(1 - z^{n_i}) = \sum_{i=0}^{n} \alpha_i z^i$, where $n = \deg f$, we define the norms

$$\| f \|_1 = \sum_{i=0}^{n} | \alpha_i |$$

$$\| f \|_2 = \left(\sum_{i=0}^{n} \alpha_i^2\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})^2 d\theta\right)^{1/2}$$

$$\| f \|_\infty = \sup_{|z|=1} | f(z) | .$$

We observe that $\| f \|_1$ is twice the size of the solution $\{\alpha_i\}$, $\{\beta_i\}$ of the Prouhet-Tarry-Escott problem.

LEMMA 5.

$$\frac{\| f \|_1}{\sqrt{\deg f + 1}} \leqslant \| f \|_2 \leqslant \| f \|_\infty \leqslant \| f \|_1 \leqslant \| f \|_2^2 .$$

Proof. This is all easily established. It all follows from well known inequalities and the fact that the coefficients of f are integers. □

In 1958 [8] Erdős and Szekeres formulated the problem of finding

$$A(k) = \min_{n_1, \ldots, n_k} \left\| \prod_{i=1}^{k} (1 - z^{n_i}) \right\|_\infty$$

They have conjectured that $A(k) \geqslant k^C$ for any C. There has been very little progress in this pretty old problem. Though an interesting and possibly related problem is solved in [2]. See Section 6.

We can use pure product solutions of the Prouhet-Tarry-Escott problem to find upper bounds for $A(k)$. These are not good general bounds, but we do find good upper bounds for small values of k using specific solutions. The following table was derived using various greedy algorithms to find the $\{n_i\}$.

k	$\|f\|_1$	$\{n_1, \ldots, n_k\}$
1	2	$\{1\}$
2	4	$\{1, 2\}$
3	6	$\{1, 2, 3\}$
4	8	$\{1, 2, 3, 4\}$
5	10	$\{1, 2, 3, 5, 7\}$
6	12	$\{1, 1, 2, 3, 4, 5\}$
7	16	$\{1, 2, 3, 4, 5, 7, 11\}$
8	16	$\{1, 2, 3, 5, 7, 8, 11, 13\}$
9	20	$\{1, 2, 3, 4, 5, 7, 9, 11, 13\}$
10	24	$\{1, 2, 3, 4, 5, 7, 9, 11, 13, 17\}$
11	28	$\{1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19\}$
12	36	$\{1, \ldots, 9, 11, 13, 17\}$
13	48	$\{1, \ldots, 9, 11, 13, 17, 19\}$
14	56	$\{1, \ldots, 7, 9, 10, 11, 13, 15, 16, 17\}$
15	60	$\{1, \ldots, 7, 9, 10, 11, 13, 15, 16, 17, 19\}$
16	60	$\{1, \ldots, 11, 13, 15, 17, 19, 23\}$
17	68	$\{1, \ldots, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29\}$
18	84	$\{1, \ldots, 11, 13, 14, 16, 17, 19, 22, 23\}$
19	100	$\{1, \ldots, 11, 13, 15, 17, 19, 21, 23, 25, 29\}$
20	116	$\{1, \ldots, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31\}$
21	130	$\{1, \ldots, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$
22	140	$\{1, \ldots, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37\}$
23	156	$\{1, \ldots, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37\}$
24	204	$\{1, \ldots, 7, 9, 10, 11, 13, 15, 16, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37\}$
25	188	$\{1, \ldots, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 41\}$
26	228	$\{1, \ldots, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41\}$
27	276	$\{1, \ldots, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41\}$
28	336	$\{1, \ldots, 13, 15, 17, 18, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41\}$
29	392	$\{1, 1, 2, 2, \ldots, 27\}$
30	432	$\{1, 1, 1, 2, \ldots, 28\}$

k	$\|f\|_1$	$\{n_1, \ldots, n_k\}$
40	1900	{1, 2, 2, ..., 17, 19, ..., 29, 31, ..., 37, 43, 47, 49, 49}
41	1348	{1, 2, 2, ..., 17, 19, ..., 29, 31, ..., 38, 40, 43, 49, 53}
42	1936	{1, 2, 2, ..., 17, 19, ..., 29, 31, ..., 38, 40, 43, 47, 52, 53}
43	2396	{1, 2, 2, ..., 17, 19, ..., 29, 31, ..., 38, 40, 43, 46, 52, 53, 60}
44	2492	{1, 2, 2, ..., 29, 31, ..., 38, 40, 43, 46, 52, 53, 60}
45	2684	{1, 2, 2, ..., 29, 31, ..., 38, 40, 43, 44, 46, 52, 53, 60}
46	2336	{1, 2, 2, ..., 29, 31, ..., 38, 40, 43, 44, 46, 48, 52, 53, 60}
47	3196	{1, 2, 2, ..., 29, 31, ..., 38, 40, 40, 43, 44, 46, 48, 52, 53, 60}
48	4080	{1, 2, 2, ..., 29, 31, ..., 38, 40, 40, 43, 44, 46, 48, 50, 52, 53, 60}
49	4086	{1, 2, 2, ..., 29, 31, ..., 38, 40, 40, 43, 44, 46, 48, 50, 52, 53, 55, 60}
50	5088	{1, 2, 2, ..., 29, 31, ..., 38, 40, 40, 43, 44, 46, 48, 49, 50, 52, 53, 55, 60}
51	5480	{1, 2, 2, ..., 29, 31, ..., 38, 40, 40, 43, 44, 46, 48, 49, 50, 52, 53, 55, 56, 60}
52	5296	{1, ..., 11, 13, 16, 17, 24, 52, ..., 56, ..., 58, 80, 82, 83, 84, 86, 88, 89, 92, 95, 100}
53	6000	{1, ..., 11, 13, 16, 17, 24, 52, 53, 54, 56, 58, ..., 80, 82, 83, 84, 86, 88, 89, 90, 92, 95, 100, 142}
54	7352	{1, 1, 2, 2, ..., 29, 31, ..., 38, 40, 42, 43, 44, 46, 48, ..., 53, 55, 56, 60}
55	5044	{1, 1, 2, 2, ..., 29, 31, ..., 38, 40, 42, 43, 44, 46, ..., 56, 60}
56	7536	{1, 1, ..., 11, 13, 16, 17, 24, 52, 53, 54, 56, 58, ..., 80, 82, ..., 92, 95, 100}
57	7156	{1, 1, ..., 11, 13, 16, 17, 24, 52, ..., 56, 58, ..., 80, 82, ..., 92, 95, 100}
58	6268	{1, 1, 2, 2, ..., 29, 31, ..., 38, 41, ..., 44, 46, ..., 60}
59	7572	{1, 1, ..., 11, 13, ..., 17, 24, 52, ..., 52, 58, ..., 80, 82, ..., 92, 95, 100}
60	10848	{1, 1, ..., 11, 13, ..., 17, 24, 52, ..., 56, 58, ..., 80, 82, ..., 92, 95, 100, 100}
80	1629900	{1, ..., 73, 90, ..., 95, 97}
100	41947220	{1, ..., 89, 107, ..., 117}

For $k = 1, 2, 3, 4, 5, 6$, and 8 these products are ideal solutions and therefore also optimal. These may well be the only k for which pure products give ideal solutions. We computed extensively on degree 6 ($k = 7$) and could not find a degree 6 product with $\|f\|_1 = 14$. Since $\|f\|_1$ is always an even integer we therefore conjecture that the minimum attainable is 16 (as above). For larger k there is no reason to believe that we have found minimal examples. This table also provides some good bounds for $N(k)$. For example $N(29) \leqslant 216$ which is much better than the bound of 419 that derives from the discussion following Proposition 3. There are many partial results on the Erdős-Szekeres problem to be found in [8], [1], [6], [14], [3], [20], [2], [16] and [13]. We give one such new result here.

We now construct an easy example to show that we cannot in general expect exponential growth of the norms of the partial products of $\prod_{i=1}^{\infty}(1-z^{\beta_i})$ on the unit disk. From this point on, $\|f\|$ without a subscript will denote $\|f\|_\infty$.

LEMMA 6. *Let* $1 \leqslant \beta_1 < \beta_2 < \ldots$ *and let*

$$W_n(z) = \prod_{1 \leqslant i < j \leqslant n} (1 - z^{\beta_j - \beta_i})$$

then

$$\| W_n(z) \| \leqslant n^{\frac{n}{2}} .$$

Proof. We can explicitly evaluate the Vandermonde determinant

$$D_n := \prod_{1 \leqslant i < j \leqslant n} (z^{\beta_j} - z^{\beta_i}) = \begin{vmatrix} 1 & z^{\beta_1} & \cdots & z^{(n-1)\beta_1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & z^{\beta_n} & \cdots & z^{(n-1)\beta_n} \end{vmatrix}$$

and by Hadamard's inequality, since each entry of the matrix has modulus at most one in the unit disk,

$$\| D_n \| \leqslant n^{n/2} .$$

Thus

$$\left\| \prod_{1 \leqslant i < j \leqslant n} (1 - z^{\beta_j - \beta_i}) \right\| = \left\| \prod_{1 \leqslant i < j \leqslant n} (z^{\beta_j} - z^{\beta_i}) \right\| \leqslant n^{n/2} . \qquad \square$$

Observe, as Dobrowolski did in [6], that if we take $\beta_i = i$, we deduce that

$$\left\| \prod_{i=1}^{n} (1 - z^i)^{n-i-1} \right\| \leqslant n^{n/2} ,$$

a result originally obtained by Atkinson in [1].

PROPOSITION 11. *Let* β_i *be the sequence formed by taking the set* $\{2^n - 2^m : n > m \geqslant 0\}$ *in increasing order. Then for all* n,

$$\left\| \prod_{i=1}^{n} (1 - z^{\beta_i}) \right\| \leqslant (32n)^{\sqrt{n/8}} .$$

Proof. Note that $2^n - 2^m \geqslant 2^m$ if $n > m$ and that $2^{n_1} - 2^{m_1} = 2^{n_2} - 2^{m_2}$ if and only if $(n_1, m_1) = (n_2, m_2)$. So whenever $n = \frac{k(k-1)}{2}$ for some k we have

$$\left\| \prod_{i=1}^{n} (1 - z^{\beta_i}) \right\| = \left\| \prod_{1 \leqslant i < j \leqslant k} (z^{2^{j-1}} - z^{2^{i-1}}) \right\| \leqslant k^{k/2} \leqslant \sqrt{2n}^{\sqrt{n/2}} .$$

While if $\frac{k(k-1)}{2} < n < \frac{(k+1)k}{2}$ then

$$\left\| \prod_{i=1}^{n} (1 - z^{\beta_i}) \right\| \leqslant \left\| \prod_{1 \leqslant i < j \leqslant k} (z^{2^{j-1}} - z^{2^{i-1}}) \right\| \left\| \prod_{i=\frac{k(k-1)}{2}+1}^{n} (1 - z^{\beta_i}) \right\|$$

$$\leqslant \sqrt{2n}^{\sqrt{n/2}} 2^{n - \frac{k(k-1)}{2} - 1} \leqslant \sqrt{2n}^{\sqrt{n/2}} 2^{k-1}$$

$$\leqslant \sqrt{2n}^{\sqrt{n/2}} 2^{\sqrt{2n}} = (32n)^{\sqrt{n/8}} . \qquad \square$$

This is not as good an estimate as Odlyzko's in [16] (see also [13]) which has exponent roughly $n^{1/3}$. What distinguishes it is that it holds for all the partial products of a single infinite product (with distinct increasing exponents). Also, clearly any $\alpha > 2$ could play the role of 2 in the construction of the β_i with the exact same conclusion.

THEOREM 1. *Let* $\{\delta_i\}$ *be any sequence of integers and let* $\{\beta_i\}$ *be the sequence of differences in the following order*

$$\{\delta_1 - \delta_0, \delta_2 - \delta_0, \delta_2 - \delta_1, \ldots, \delta_n - \delta_0, \ldots, \delta_n - \delta_{n-1}, \ldots\}$$

then

$$\left\| \prod_{i=1}^{n} (1 - z^{\beta_i}) \right\| \leqslant (32n)^{\sqrt{n/8}} .$$

5. PERFECT SOLUTIONS OF PRIME SIZE

The first unresolved case of the Prouhet-Tarry-Escott problem is the eleven case. The previous ideal solutions were all found without computer assistance; indeed the cases 1, ..., 10 were all resolved prior to 1950. It therefore seems appropriate to discuss an algorithm for searching for such solutions. We wish to perform a computer search for perfect symmetric ideal solutions