Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 40 (1994)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Rubrik: COMMISSION INTERNATIONALE DE L'ENSEIGNEMENT MATHÉMATIQUE (THE INTERNATIONAL COMMISSION ON MATHEMATICAL INSTRUCTION)

L'Enseignement Mathématique, t. 40 (1994), p. 345-357

COMMISSION INTERNATIONALE DE L'ENSEIGNEMENT MATHÉMATIQUE

(THE INTERNATIONAL COMMISSION ON MATHEMATICAL INSTRUCTION)

PERSPECTIVES ON THE TEACHING OF GEOMETRY FOR THE 21st CENTURY

DISCUSSION DOCUMENT FOR AN ICMI STUDY

1. WHY A STUDY ON GEOMETRY?

Geometry, considered as a tool for understanding, describing and interacting with the space in which we live, is perhaps the most intuitive, concrete and reality-linked part of mathematics. On the other hand geometry, as a discipline, rests on an extensive formalization process, which has been carried out for over 2000 years at increasing levels of rigour, abstraction and generality.

In recent years, research in geometry has been greatly stimulated by new ideas both from inside mathematics and from other disciplines, including computer science. At present, the enormous possibilities of computer graphics influence many aspects of our lives; in order to use these possibilities, a suitable visual education is needed.

Among mathematicians and mathematics educators there is a widespread agreement that, due to the manifold aspects of geometry, the teaching of geometry should start at an early age, and continue in appropriate forms throughout the whole mathematics curriculum. However, as soon as one tries to enter into details, opinions diverge on how to accomplish the task. There have been in the past (and there persist even now) strong disagreements about the aims, contents and methods for the teaching of geometry at various levels, from primary school to university.

Perhaps one of the main reasons for this situation is that geometry has so many aspects, and as a consequence there has not yet been found — and perhaps there does not exist at all — a simple, clean, linear "hierarchical" path from the first beginnings to the more advanced achievements of geometry. Unlike what happens in arithmetic and algebra, even basic concepts in geometry, such as the notions of angle and distance, have to be reconsidered at different stages from different viewpoints.

Another problematic point concerns the role of proofs in geometry: relations between intuition, inductive and deductive proofs, age of students at which proofs can be introduced, and different levels of rigour and abstraction.

Thus the teaching of geometry is not at all an easy task. But instead of trying to face and overcome the obstacles arising in the teaching of geometry, actual school-practice in many countries has simply bypassed these obstacles, cutting out the more demanding parts, often without any replacement. For instance, three-dimensional geometry has almost disappeared or has been confined to a marginal role in the curricula in most countries.

Starting from this analysis, and specifically considering the gap between the increasing importance of geometry for its own sake, as well as in research and in society, and the decline of its role in school curricula, ICMI feels that there is an urgent need for an international study, whose main aims are:

— To discuss the goals of the teaching of geometry at different school levels and according to different cultural traditions and environments.

— To identify important challenges and emerging trends for the future and to analyze their potential didactical impact.

— To exploit and implement new teaching methods.

2. Aspects of geometry

The outstanding historical importance of geometry in the past, in particular as a prototype of an axiomatic theory, is so universally acknowledged that it deserves no further comment. Moreover, in the last century and specifically during the last decades, as Jean Dieudonné asserted at ICME 4 (Berkeley, 1980), Geometry "bursting out of its traditional narrow confines [...] has revealed its hidden powers and its extraordinary versatility and adaptability, thus becoming one of the most universal and useful tools in all parts of mathematics" (J. Dieudonné: The Universal Domination of Geometry, ZDM 13 (1), p. 5-7 (1981)).

Actually, geometry includes so many different aspects, that it is hopeless (and maybe even useless) to write out a complete list of them. Here we mention only those aspects, which in our opinion are particularly relevant in view of their didactical implications:

— Geometry, as *the science of space*. From its roots as a tool for describing and measuring figures, geometry has grown into a theory of ideas and methods by which we can construct and study idealized models of the physical world as well as of other real world phenomena. According to different points of view, we get euclidean, affine, descriptive, projective geometry, but also topology or non-euclidean and combinatorial geometries.

— Geometry as *a method for visual representations of concepts and processes* from other areas in mathematics and in other sciences; e.g. graphs and graph theory, diagrams of various kinds, histograms.

— Geometry as *a meeting point* between mathematics as a theory and mathematics as a model resource.

— Geometry as *a way of thinkings and understanding* and, at a higher level, as *a formal theory.*

— Geometry as *a paradigmatic example for teaching deductive reasoning.*

— Geometry as *a tool in applications,* both traditional and innovative. The latter ones include e.g. computer graphics, image processing and image manipulation, pattern recognition, robotics, operations research.

Another distinction should be made with respect to several different approaches according to which one may deal with geometry. Roughly speaking, possible approaches are:

- manipulative;
- intuitive;
- deductive;
- analytic.

Also one may distinguish between a geometry which stresses "static" properties of geometric objects and a geometry where objects are considered in a "dynamic" setting, as they change under the effect of different types of space transformations.

3. Is there a crisis in the teaching of geometry?

During the second half of this century geometry seems to have progressively lost its former central position in mathematics teaching in most countries. The decrease has been both qualitative and quantitative. Symptoms of this decrease may be found for instance in recent national and international surveys on the mathematical knowledge of students. Often geometry is totally ignored or only a very few items concerned with geometry are included. In the latter case questions tend to be confined to some elementary "facts" about simple figures and their properties, and performance is reported to be relatively poor.

What are the main causes of this situation?

— From about 1960 to 1980 a general time pressure on traditional topics has occurred, due to the introduction of new topics in mathematics curricula (e.g. probability, statistics, computer science, discrete mathematics). At the same time the number of school hours devoted to mathematics has gone down. The "modern mathematics movement" has contributed — at least indirectly — to the decline of the role of euclidean geometry, favouring other aspects of mathematics and other points of view for its teaching (e.g. set theory, logic, abstract structures). The decline has involved in particular the role of visual aspects of geometry, both three-dimensional and two-dimensional, and all those parts which did not fit into a theory of linear spaces as, for instance, the study of conic sections and of other noteworthy curves.

— In more recent years there has been a shift back towards more traditional contents in mathematics, with a specific emphasis on problem posing and problem solving activities. However, attempts to restore classical euclidean geometry — which earlier in many parts of the world was the main subject in school geometry — have so far not been very successful. The point is that in traditional courses on euclidean geometry the material is usually presented to students as a ready-made end product of mathematical activity. Hence, in this form, it does not fit well into curricula where pupils are expected to take an active part in the development of their mathematical knowledge.

— In most countries the percentage of young people attending secondary school has increased very rapidly during the last decades. Thus the traditional way of teaching abstract geometry to a selected minority has become both more difficult and more inappropriate for the expectations of the majority of students of the new generations. At the same time, the need for more teachers has caused, on average, a decline in their university preparation, especially with respect to the more demanding parts of mathematics, in particular geometry. Since younger teachers have learned mathematics under curricula that neglected geometry, they lack a good background in this field, which in turn fosters in them the tendency to neglect the teaching of geometry to their pupils.

The situation is even more dramatic in those countries which lack a prior tradition in schooling. In some cases geometry is completely absent from their mathematics curricula.

— The gap between the conception of geometry as a research area and as a subject to be taught in schools seems to be increasing; but so far no consensus has been found on how to bridge this gap, nor even whether it could (or should) be bridged through an introduction of more advanced topics in school curricula at lower grades.

4. Geometry as reflected in education

In former sections, we have considered geometry mainly as a mathematical theory and have analyzed some aspects of its *teaching*. Since *learning* is unquestionably the other essential pole of any educational project, it is now appropriate to pay due attention to the main variables which may affect a coherent teaching/learning process. Consequently, several different aspects or "dimensions" (considered in their broadest meaning) must be taken into account:

— *The social dimension,* with two poles:

- The cultural pole, i.e. the construction of a common background (knowledge and language) for all the people sharing a common civilization;
- The educational pole, i.e. the development of criteria, internal to each individual, for self consistency and responsibility.

— *The cognitive dimension,* i.e. the process which, starting from reality, leads gradually to a refined perception of space.

— *The epistemological dimension,* i.e. the ability to exploit the interplay between reality and theory through modelling (make previsions, evaluate their effects, reconsider choices). Thereby axiomatization enables one to get free from reality; this in turn may be seen as a side-step which allows further conceptualization.

— *The didactic dimension,* i.e. the relation between teaching and learning. Within this dimension several aspects deserve consideration. As an example, we list three of them:

- To make various fields interact (both within mathematics and between mathematics and other sciences).
- To make sure that the viewpoints of the teacher and the pupils are consistent in a given study. For instance, to be aware that different distance scales may involve different conceptions and processes adopted by the pupils, even though the mathematical situation is the same: in a "space of small objects", visual perception may help to make conjectures and to identify geometric properties; when dealing with the space where we are used to move around (the classroom, for instance) it is still easy to get local information, but it may be difficult to achieve an overall view; in a "large scale space" (as is the case in geography or in astronomy) symbolic representations are needed in order to analyze its properties.
- To pay due consideration to the influence of tools available in teaching/learning situations (from straightedge and compass, as well as other concrete materials, to graphic calculators, computers and specific software).

It goes without saying that all these dimensions are interrelated with each other and that they should also be related appropriately to different age levels and school types: pre-primary level, primary level, lower secondary level, upper secondary level (where differentiation into academic, technical, vocational tracks usually starts), tertiary (i.e. university) level, including teacher preparation.

5. New technology and teaching aids for geometry

There is a long tradition of mathematicians making use of technological tools, and conversely the use of these tools has given rise to many challenging mathematical problems (e.g. straightedge and compass for geometric constructions, logarithms and mechanical instruments for numerical computations). In recent years new technology, and in particular computers, has affected dramatically all aspects of our society. Many traditional activities have become obsolete, while new professions and new challenges arise. For instance, technical drawing is no longer done by hand. Nowadays, instead, one uses commercial software, plotters and other technological devices. CAD/CAM and symbolic algebra software are becoming widely available.

Computers have also made it possible to construct "virtual realities" and to generate interactively animations or marvellous pictures (e.g. fractal images). Moreover, electronic devices can be used to achieve experiences that in everyday life are either inaccessible, or accessible only as a resut of time-consuming and often tedious work.

Of course, in all these activities geometry is deeply involved, both in order to enhance the ability to use technological tools appropriately, and in order to interpret and understand the meaning of the images produced.

Computers can be used also to gain a deeper understanding of geometric structures thanks to software specifically designed for didactical purposes. Examples include the possibility of simulating traditional straightedge and compass constructions, or the possibility of moving basic elements of a configuration on the screen while keeping existing geometric relationships fixed, which may lead to a dynamic presentation of geometric objects and may favour the identification of their invariants.

Until now, school practice has been only marginally influenced by these innovations. But in the near future it is likely that at least some of these new topics will find their way into curricula. This will imply great challenges:

- How will the use of computers affect the teaching of geometry, its aims, its contents and its methods?
- Will the cultural values of classical geometry thereby be preserved, or will they evolve, and how?
- In countries where financial constraints will not allow a massive introduction of computers into schools in the near future, will it nevertheless be possible to restructure geometry curricula in order to cope with the main challenges originated by these technological devices?

6. Key issues and challenges for the future

In this section we list explicitly some of the most relevant questions which arise from the considerations outlined in the preceding sections. We believe that a clarification of these issues would contribute to a significant improvement in the teaching of geometry. Of course we do not claim that all the problems quoted below are solvable, nor that the solutions are unique and have universal validity. On the contrary, the solutions may vary according to different school levels, different school types and different cultural environments.

6.1. Aims

Why is it advisable and/or necessary to teach geometry?

Which of the following may be considered to be the most relevant aims of the teaching of geometry?

— To describe, understand and interpret the real world and its phenomena.

— To supply an example of an axiomatic theory.

— To provide a rich and varied collection of problems and exercises for individual student activity.

— To train learners to make guesses, state conjectures, provide proofs, and find out examples and counterexamples.

— To serve as a tool for other areas of mathematics.

— To enrich the public perception of mathematics.

6.2. Contents

What should be taught?

Is it preferable to emphasize "breadth" or "depth" in the teaching of geometry? And is it possible/advisable to identify a core curriculum?

In the case of an affirmative answer to the second question above, which topics should be included in syllabi at various school levels?

In the case of a negative answer, why is it believed that teachers or local authorities should be left free to choose the geometric contents according to their personal tastes (is this point of view common to other mathematical subjects, or is it peculiar to geometry)?

Should geometry be taught as a specific, separate subject, or should it be merged into general mathematical courses?

There seems to be widespread agreement that the teaching of geometry must reflect the actual and potential needs of society. In particular, geometry of three-dimensional space should be stressed at all school levels, as well as the relationships between three-dimensional and two-dimensional geometry. How could and should the present situation (where only two-dimensional geometry is favoured) therefore be modified and improved?

In which ways can the study of linear algebra enhance the understanding of geometry? At what stage should "abstract" vector space structures be introduced? And what are the goals?

Would it be possible and advisable also to include some elements of non-euclidean geometries into curricula?

6.3. Methods

How should we teach geometry?

Any topic taught in geometry can be located somewhere between the two extremes of an "intuitive" approach and a "formalized" or "axiomatic" approach. Should only one of these two approaches be stressed at each school level, or should there be a dialectic interplay between them, or else should there be a gradual shift from the former to the latter one, as the age of students and the school level progresses?

What is the role of axiomatics within the teaching of geometry? Should a complete set of axioms be stated from the beginning (and, if so, at what age and school level) or is it advisable to introduce axiomatics gradually, e.g. via a "local deductions" method?

Traditionally, geometry is the subject where "one proves theorems". Should "theorem proving" be confined to geometry?

Would we like to expose students to different levels of rigour in proofs (as age and school level progress)? Should proofs be tools for personal understanding, for convincing others, or for explaining, enlightening, verifying?

Starting from a certain school level, should every statement be proved, or should only a few theorems be selected for proof? In the latter case, should one choose these theorems because of their importance within a specific theoretical framework, or because of the degree of difficulty of their proof? And should intuitive or counterintuitive statements be privileged?

It seems that there is an international trend towards the teaching of analytic methods in increasingly earlier grades, at the expense of other (synthetic) aspects of geometry. Analytic geometry is supposed to present algebraic models for geometric situations. But, as soon as students are introduced to these new methods, they are suddenly projected into a new world of symbols and calculations in which the link between geometric situations and their algebraic models breaks down and geometric interpretations of numerical calculations are often neglected. Hence, at what age and school level should teaching of analytic geometry start? Which activities, methods and theoretical frameworks can be used in order to restore the link between the algebraic representation of space and the geometric situation it symbolizes?

How can we best improve the ability of pupils to choose the appropriate tools for solving specific geometric problems (conceptual, manipulative, technological)?

6.4. Books, computers, and other teaching aids

Are traditional textbooks as appropriate for teaching and learning geometry as we would like them to be?

How do teachers and pupils actually use geometry textbooks and other teaching aids? How would we like pupils to use them?

What changes could and should be made in teaching and learning geometry in the perspective of increased access to software, videos, concrete materials and other technological devices?

What are the advantages, from the educational and geometrical point of view, that can follow from the use of such tools?

Which problems and limitations may arise from the use of such tools, and how can they be overcome?

To what extent is knowledge acquired in a computer environment transferable to other environments?

6.5. Assessment

The ways of assessment and evaluation of pupils strongly influence teaching and learning strategies. How should we set out objectives and aims, and how should we construct assessment techniques that are consistent with these objectives and aims? Are there issues of assessment which are peculiar to the teaching and learning of geometry?

How does the use of calculators, computers and specific geometric software influence examinations as regards content, organization and criteria for the evaluation of the answers of the students?

Should assessment procedures be based mainly upon written examination papers (as it seems to be customary in many countries) or else what should be the role of oral communication, of technical drawing and of work with the computer?

What is it exactly that should be evaluated and considered for assessment: The solution outcome? The solution process? The method of thinking? Geometric constructions?

6.6. Teacher preparation

One essential component of an efficient teaching/learning process is good teacher preparation, as regards both disciplinary competence and educational, epistemological, technological and social aspects. Hence, what specific preparation in geometry is needed (and realistically achievable) for prospective and practicing teachers?

It is well known that teachers tend to reproduce in their profession the same models they experienced when they were students, regardless of subsequent exposure to different points of view. How is it then possible to motivate the need for changes in the perspective of teaching geometry (both from the content and from the methodological point of view)?

Which teaching supplies (books, videos, software, ...) should be made available for in-service training of teachers, in order to favour a flexible and open-minded approach to the teaching of geometry?

6.7. Evaluation of long-term effects

All too often the success (or failure) of a curricular and/or methodological reform or innovation for a certain school system is evaluated on the basis only of a short period of observation of its outcomes. Moreover usually there are no comparative studies on the possible side effects of a change of content or methods. Conversely, it would be necessary to look also at what happens in the long term. For instance:

— Does a visual education from a very young age have an impact on geometric thinking at a later stage?

— How does an early introduction of analytic methods in the teaching of geometry influence the visual intuition of pupils? When these pupils become professionals, do they rely more on the intuitive or on the rational parts of the geometry teaching to which they have been exposed?

— What is the impact of an extensive use of technological tools on geometry learning?

6.8. Implementation

At ICME 5 (Adelaide, 1984) J. Kilpatrick asked a provocative question: What do we know about mathematics education in 1984 that we did not know in 1980? Recently the same question has been picked up again in the ongoing ICMI study: "What is research in mathematics education, and what are its results". As for geometry, the possibility of relying on research results would be extremely useful in order to avoid reproposing in the future paths already proved unsuccessful, and conversely in order to benefit from successful solutions. And, as for still unsettled and relevant questions, we would like research to give us useful information in order to clarify the advantages and drawbacks of possible alternatives.

Hence, a key question might be:

What do we already know from research about the teaching and learning of geometry and what would we want future research to tell us?

7. Call for papers

The ICMI study "Perspectives on the Teaching of Geometry for the 21st Century" will consist of an invited *Study Conference* and a *Publication* to appear in the ICMI study series, based on the contributions to, and the outcomes of, the Conference.

The Conference is scheduled for September 1995 in Catania (Italy). The International Program Committee (IPC) for the study hereby invites individuals and groups to submit ideas, suggestions and contributions on major problems or issues related to this discussion document, no later than February 15, 1995.

Although participation in the conference requires an invitation from the IPC, "experts" and "newcomers" interested in contributing to and participating in the conference are encouraged to contact the chair of the IPC. Unfortunately, an invitation to attend does not imply that financial support will be provided by the organizers.

Papers, as well as suggestions concerning the content of the study conference program should be sent to

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