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The algebra of (iii) contains the  $f_i$ 's by definition. That (iii) implies (i) will follow if we can show that any element of  $\cup_{t < n} \mathcal{A}(n, n; t)$  is expressible as a product of  $F_i$ 's. That this is true for diagrams having a straight through-string is a well known fact about the Temperley-Lieb algebra. But if  $D$  is an oriented diagram with less than  $n$  through-strings, either  $D$  has zero through-string and we are in the Temperley-Lieb situation, or  $D \circ u^k$  has a straight through-string for some even  $k$ . Thus  $Du^k$  is a word on the  $F_i$ 's and it suffices to show that  $F_i u^2$  is a word on the  $F_i$ 's for all  $i$ . It follows from a picture that  $F_i u^{-2} = F_i F_{i+1} \dots F_n F_1 F_2 \dots F_{i-2}$ .  $\square$

*Remark 2.9.* We leave it to the reader to show that Lemma 2.8 is true without the  $\rightarrow$ 's if  $n$  is odd.

*Remark 2.10.* It follows from 2.8 that the elements  $v_t$  are in the algebra generated by the  $F_i$ 's for  $t < n$ . We record the expression

$$v_{n-2}^2 = F_n \circ F_1 \circ F_2 \circ \dots \circ F_n .$$

Thus rotations are unavoidable even if one is only interested in the structure of the algebra generated by the  $F_i$ 's.

### 3. THE BRAUER REPRESENTATION

So far we have begged the important question of when the algebra  $A(n, \delta)$  is semisimple. We do not have a complete answer for this but we shall show that it is semisimple whenever  $\delta$  is an integer  $\geq 3$ , (and that  $A(n, -2)$  is not semisimple for  $n \geq 3$ ) by using a representation onto a  $C^*$ -algebra which we will show to be faithful for such  $\delta$ . That the representation is faithful for  $n$  fixed and large integral (hence any large)  $\delta$  is rather easy.

*Definition 3.1.* Let  $V$  be a vector space of dimension  $k$  and basis  $w_1, w_2, \dots, w_k$ . If the diagram  $D \in D(n, n)$  has  $n$  connecting edges called  $\varepsilon$ , define  $\beta(D) \in \text{End}(\otimes^n V)$  by the matrix (with respect to the basis  $\{w_{a_1} \otimes w_{a_2} \otimes \dots \otimes w_{a_n} \mid a_i = 1, 2, \dots, k\}$  of  $\otimes^n V$ )

$$\beta(D)_{a_1 a_2 \dots a_n}^{a_{n+1} \dots a_{2n}} = \prod_{\varepsilon} \delta(a_{s(\varepsilon)}, a_{b(\varepsilon)})$$

where  $s(\varepsilon), b(\varepsilon)$  are the two ends of the edge  $\varepsilon$ , labelled from 1 to  $2n$ , and, just in this formula,  $\delta$  is the Kronecker  $\delta$ .

**LEMMA 3.2.**  $D \mapsto \beta(D)$  defines a homomorphism of  $B(n, k)$  (hence  $A(n, k)$ ) onto a  $C^*$ -subalgebra of  $\text{End}(\otimes^n V)$ .

*Proof.* This is just the orthogonal case of [B]. The  $C^*$ -structure is that for which  $V$  is a Hilbert space with orthonormal basis  $\{w_i\}$ , and it is clear that the adjoint of  $D$  is just  $D$  read backwards.

*Remark 3.3.* Since finite-dimensional  $C^*$ -algebras are semisimple, this proves that  $\beta(B(n, k))$  is always semisimple. Further note that  $\beta(A(n, k))$  is also a  $C^*$ -algebra.

**THEOREM 3.4.** *For  $k \geq 2$ ,  $\beta$  restricted to  $TL(n, k)$  is faithful for all  $n$ .*

*Proof.* The normalized trace on  $\text{End}(\otimes^n V)$  defines a Markov trace on  $TL(n, k)$  with Markov parameter  $k^2$ . Thus by the calculation of [J] or [GHJ], the structure of  $\beta(TL(n, k))$  is known and it has the same dimension as  $TL(n, k)$ .

**THEOREM 3.5.** *For  $k \geq 3$ ,  $\beta$  restricted to  $A(n, k)$  is faithful for all  $n$ .*

*Proof.* Let  $x = \sum_{D \in \mathcal{A}(n, n)} \lambda_D D$  ( $\lambda_D \in \mathbf{C}$ ) be such that  $\beta(x) = 0$ . We have seen that  $\mathcal{A}(n, n; 0)$  actually consists of planar diagrams so by 3.4 we may suppose that  $\lambda_D \neq 0$  for some  $D \in \mathcal{A}(n, n; t)$ ,  $t \geq 1$ . Thus by pre- and post-multiplying  $x$  by suitable powers of  $u$ , we may assume  $\lambda_D \neq 0$  for some  $D$  with a straight line joining the inner and outer  $*$ 's. Now split  $V$  as  $\mathbf{C}w_1 \oplus w_1^\perp$ . Since  $\dim V > 2$ ,  $\dim w_1^\perp \geq 2$ . Let  $P$  be orthogonal projection from  $\otimes^n V$  onto  $w_1 \otimes (\otimes^{n-1} w_1^\perp)$ . If  $D$  is a diagram with the inner and outer  $*$ 's not connected,  $P\beta(D)P = 0$ . Also, the set of diagrams with a straight line between the  $*$ 's is in obvious bijection with  $\mathcal{P}(n-1, n-1)$ . Thus  $0 = P\beta(x)P = \sum_{D \in \mathcal{P}(n-1, n-1)} \lambda_D P\beta(D)P$  and not all the  $\lambda_D$ 's are zero.

But the matrix of  $P\beta(D)P$  with respect to the basis

$$\{w_1 \otimes (w_{a_1} \otimes \cdots \otimes w_{a_{n-1}}) \mid a_i = 2, 3, \dots, k\}$$

is clearly that of " $\beta(D)$ " for parameters  $k-1$  and  $n-1$ . By 3.4 we conclude  $\sum_{D \in \mathcal{P}(n-1, n-1)} \lambda_D D = 0$ , a contradiction.  $\square$

**COROLLARY 3.6.**  $\mathcal{A}(n, k)$  and  $\mathcal{A}(n, k)$  are semisimple for  $k$  an integer  $\geq 3$ .

The question naturally arises of finding those values of  $\delta$  and  $n$  for which  $\mathcal{A}(n, \delta)$  is semisimple. We observe that for  $\delta = -2$ , the algebra  $\mathcal{A}(n, \delta)$  is not semisimple for  $n > 2$ . This is because we may use the Brauer representation corresponding to the symplectic case.

Then  $\beta(f_1)$  is represented on  $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$

$$\text{by the matrix } \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \text{id}$$

using a symplectic basis of  $\mathbf{C}^2$ , and  $\beta(u)$  is the obvious cyclic permutation on  $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ . But then  $2 - \beta(f_1)$  is the transposition on  $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$  exchanging the first two copies of  $\mathbf{C}^2$ . Thus the image of  $\mathcal{A}(n, \delta)$  is the same as that of the group algebra of the symmetric group.

#### 4. THE CYLINDRICAL TRACE

There is a natural trace functional  $\text{tr}$  on  $A(n, \delta)$  defined by  $\text{tr}(D) = \delta^{n(D)}$ ,  $n(D)$  being the number of closed loops formed on the cylinder if the inside and outside boundaries of the annulus are identified. We will call this trace the cylindrical trace.

*Note 4.1.* This trace exists in fact on the whole Brauer algebra — it could be defined in terms of partitions as  $\text{tr}(D) = \delta^{n(D)}$  where  $n(D)$  is the number of equivalence classes for the equivalence relation generated by  $D$  itself and the relation which identifies each point on the top with the corresponding point on the bottom.

*Note 4.2.* One has the relation  $n(D_1 \circ D_2) = n(D_2 \circ D_1)$  so one might try to define a more general trace by replacing  $\delta$  by an arbitrary complex number. But  $n(\alpha, \beta) \neq n(\beta, \alpha)$  in general so one is forced to choose  $\delta$ .

If  $\delta$  is a value for which  $A(n, \delta)$  is semisimple we know that  $A(n, \delta)$  is a direct sum of matrix algebras, so our cylindrical trace is determined by its value on a minimal idempotent in each matrix algebra summand. We will calculate these “weights” of the trace. In order to do this we will need detailed information on the multiplicities of  $u$  in each irreducible representation of  $A(n, \delta)$ .

*Definition 4.3.* For  $n \geq t > 0$  the group  $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/t\mathbf{Z} (= \{(a, b) \mid a = 0, \dots, n - 1; b = 0, \dots, t - 1\})$  acts by linear transformations on  $\mathcal{A}(t, n; t)$  by  $(a, b)(D) = u^a \circ D \circ u^b$ . (The  $u$ 's on the left and right in this formula are of course different if  $n \neq t$ .) Let  $F_{n,t}(a, b)$  be the number of fixed points for  $(a, b)$ . Let  $F_n(a)$  be the number of fixed points for the action  $D \mapsto D \circ u^a$  of  $a \in \mathbf{Z}/n\mathbf{Z}$  on  $\mathcal{A}(n, 0)$ .