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THE PROUHET-TARRY-ESCOTT PROBLEM REVISITED

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ABSTRACT. The old problem of Prouhet, Tarry, Escott and others asks one to find two distinct sets of integers $\{\alpha_1, \ldots, \alpha_n\}$, and $\{\beta_1, \ldots, \beta_n\}$ with

$$\alpha_1^m + \cdots + \alpha_n^m = \beta_1^m + \cdots + \beta_n^m$$

for $m = 1, \ldots, k$ (with the most interesting case being $k = n - 1$). We review some elementary properties of solutions and examine the fine structure of 'ideal' and 'symmetric ideal' solutions. The relationship of this problem to the 'easier' Waring problem and a problem of Erdős and Szekeres of minimizing the norm of a product of cyclotomic polynomials on the unit disk is then discussed. We present some new bounds for this problem and for the Prouhet-Tarry-Escott problem of small size. We also present an algorithm for calculating symmetric ideal p-adic solutions of the the Prouhet-Tarry-Escott problem.

1. INTRODUCTION

A classic problem in Diophantine Analysis that occurs in many guises is the Prouhet-Tarry-Escott problem. This is the problem of finding two distinct sets of integers $\{\alpha_1, \ldots, \alpha_n\}$, $\{\beta_1, \ldots, \beta_n\}$ such that

$$\begin{aligned}
\alpha_1 + \cdots + \alpha_n &= \beta_1 + \cdots + \beta_n \\
\alpha_1^2 + \cdots + \alpha_n^2 &= \beta_1^2 + \cdots + \beta_n^2 \\
&\vdots \\
\alpha_1^k + \cdots + \alpha_n^k &= \beta_1^k + \cdots + \beta_n^k .
\end{aligned}$$

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This we will call the Prouhet-Tarry-Escott Problem. We call n the size of the solution and k the degree. We abbreviate the above system by writing $\{\alpha_i\} \stackrel{k}{=} \{\beta_i\}$ and reserve α_i and β_i as integer variables.

This problem has a long history and is, in some form, over 200 years old. In 1750-51 Euler and Goldbach noted that

$$\{a, b, c, a + b + c\} \stackrel{2}{=} \{a + b, a + c, b + c\} .$$

A general solution of the problem for all degrees, but large sizes, came a century later in 1851 when Prouhet found that there are n^{k+1} numbers separable into n sets so that each pair of sets forms a solution of degree k and size n^k. Over the next 60 years some more parametric and specific solutions of degrees two, three, four and five were found. In the 1910's Tarry and Escott looked more closely at the problem and subsequently their names were attached to it. They found many specific solutions and provided a number of elementary general results. Prouhet's result, while the first general solution of the problem, was not properly noticed until 1959 when Wright [23] took exception to the problem being called the Tarry-Escott problem and drew attention to Prouhet's contribution in a paper called *Prouhet's 1851 Solution of the Tarry-Escott Problem of 1910.* More of the early history of the problem can be found in Dickson [5], where he refers to it as the problem of 'equal sums of like powers'.

The problem is called the problem of Prouhet and Tarry by Hua in his text [11], which is a good source of some of the elementary material. It has also been referred to as the Tarry problem. A good introductory paper [7] by Dorwart and Brown calls it the Tarry-Escott problem. Solutions are often called 'multigrades' as in Smyth [19].

While the Prouhet-Tarry-Escott problem is old it appears to have received only a little serious computational attention. So one particular aim is to provide some numerical insights and report the results of various computations. We computed extensively on the size 7 and size 11 cases of the problem. Eleven is of particular interest because it is the first unresolved case and we found that "no symmetric ideal" solutions exist with all $\{\alpha_i\}$ and $\{\beta_i\}$ of relatively small size ($\leqslant 363$). This is discussed in Section 5 and an algorithm is presented.

We also computed extensively on an old and related problem of Erdős and Szekeres that concerns the norms of products of cyclotomic polynomials. This is discussed and many new bounds for small sizes are given in section 4.2.

Section 2 of this paper collects together some of the elementary theory.

Section 3 then focuses on the most interesting minimal case of $n = k + 1$. The known solutions are presented and Smyth's attractive recent treatment of the largest known case ($n = 10$) is discussed. In these minimal cases a solution must have considerable additional structure.

Two related problems are discussed in Section 4. One is due to Erdős and Szekeres the other due to Wright. Both have been open for decades.

Section 6 presents some of the many open problems directly related to these matters.

2. Elementary Properties

The problem can be stated in three equivalent ways. This is an old result as are most of the results of this section in some form or another. (See for example [7], [11].) In various contexts it is easier to use different forms of the problem.

PROPOSITION 1. *The following are equivalent:*

(1) $$\sum_{i=1}^{n} \alpha_i^j = \sum_{i=1}^{n} \beta_i^j \quad \text{for} \quad j = 1, \ldots, k$$

(2) $$\deg\left(\prod_{i=1}^{n} (x - \alpha_i) - \prod_{i=1}^{n} (x - \beta_i)\right) \leqslant n - (k+1)$$

(3) $$(x-1)^{k+1} \,\Big|\, \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i}.$$

Proof. An application of Newton's symmetric polynomial identities shows the equivalence of (1) and (2). To prove the equivalence of (1) and (3) apply xd/dx to equation (3) and evaluate at one $k + 1$ times. □

A solution of the Prouhet-Tarry-Escott problem generates a family of solutions by the following lemma. Any solutions that can be derived from each other in this manner are said to be equivalent.

LEMMA 1. *If* $\{\alpha_1, \ldots, \alpha_n\}, \{\beta_1, \ldots, \beta_n\}$ *is a solution of degree* k, *then so is* $\{M\alpha_1 + K, \ldots, M\alpha_n + K\}, \{M\beta_1 + K, \ldots, M\beta_n + K\}$ *for arbitrary integers* M, K.

Proof. The second form of the problem is clearly preserved when the polynomials $\prod_{i=1}^{n}(x - \alpha_i)$ and $\prod_{i=1}^{n}(x - \beta_i)$ are scaled and translated by integer constants. □