

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 40 (1994)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE PROUHET-TARRY-ESCOTT PROBLEM REVISITED  
**Autor:** Borwein, Peter / Ingalls, Colin  
**Kapitel:** 1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-61102>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## THE PROUHET-TARRY-ESCOTT PROBLEM REVISITED

by Peter BORWEIN and Colin INGALLS

ABSTRACT. The old problem of Prouhet, Tarry, Escott and others asks one to find two distinct sets of integers  $\{\alpha_1, \dots, \alpha_n\}$ , and  $\{\beta_1, \dots, \beta_n\}$  with

$$\alpha_1^m + \dots + \alpha_n^m = \beta_1^m + \dots + \beta_n^m$$

for  $m = 1, \dots, k$  (with the most interesting case being  $k = n - 1$ ). We review some elementary properties of solutions and examine the fine structure of 'ideal' and 'symmetric ideal' solutions. The relationship of this problem to the 'easier' Waring problem and a problem of Erdős and Szekeres of minimizing the norm of a product of cyclotomic polynomials on the unit disk is then discussed. We present some new bounds for this problem and for the Prouhet-Tarry-Escott problem of small size. We also present an algorithm for calculating symmetric ideal  $p$ -adic solutions of the the Prouhet-Tarry-Escott problem.

### 1. INTRODUCTION

A classic problem in Diophantine Analysis that occurs in many guises is the Prouhet-Tarry-Escott problem. This is the problem of finding two distinct sets of integers  $\{\alpha_1, \dots, \alpha_n\}$ ,  $\{\beta_1, \dots, \beta_n\}$  such that

$$\begin{aligned}\alpha_1 + \dots + \alpha_n &= \beta_1 + \dots + \beta_n \\ \alpha_1^2 + \dots + \alpha_n^2 &= \beta_1^2 + \dots + \beta_n^2 \\ &\vdots \quad \vdots \quad \vdots \\ \alpha_1^k + \dots + \alpha_n^k &= \beta_1^k + \dots + \beta_n^k.\end{aligned}$$

This we will call the Prouhet-Tarry-Escott Problem. We call  $n$  the size of the solution and  $k$  the degree. We abbreviate the above system by writing  $\{\alpha_i\} \stackrel{k}{=} \{\beta_i\}$  and reserve  $\alpha_i$  and  $\beta_i$  as integer variables.

This problem has a long history and is, in some form, over 200 years old. In 1750-51 Euler and Goldbach noted that

$$\{a, b, c, a + b + c\} \stackrel{2}{=} \{a + b, a + c, b + c\}.$$

A general solution of the problem for all degrees, but large sizes, came a century later in 1851 when Prouhet found that there are  $n^{k+1}$  numbers separable into  $n$  sets so that each pair of sets forms a solution of degree  $k$  and size  $n^k$ . Over the next 60 years some more parametric and specific solutions of degrees two, three, four and five were found. In the 1910's Tarry and Escott looked more closely at the problem and subsequently their names were attached to it. They found many specific solutions and provided a number of elementary general results. Prouhet's result, while the first general solution of the problem, was not properly noticed until 1959 when Wright [23] took exception to the problem being called the Tarry-Escott problem and drew attention to Prouhet's contribution in a paper called *Prouhet's 1851 Solution of the Tarry-Escott Problem of 1910*. More of the early history of the problem can be found in Dickson [5], where he refers to it as the problem of 'equal sums of like powers'.

The problem is called the problem of Prouhet and Tarry by Hua in his text [11], which is a good source of some of the elementary material. It has also been referred to as the Tarry problem. A good introductory paper [7] by Dorwart and Brown calls it the Tarry-Escott problem. Solutions are often called 'multigrades' as in Smyth [19].

While the Prouhet-Tarry-Escott problem is old it appears to have received only a little serious computational attention. So one particular aim is to provide some numerical insights and report the results of various computations. We computed extensively on the size 7 and size 11 cases of the problem. Eleven is of particular interest because it is the first unresolved case and we found that "no symmetric ideal" solutions exist with all  $\{\alpha_i\}$  and  $\{\beta_i\}$  of relatively small size ( $\leq 363$ ). This is discussed in Section 5 and an algorithm is presented.

We also computed extensively on an old and related problem of Erdős and Szekeres that concerns the norms of products of cyclotomic polynomials. This is discussed and many new bounds for small sizes are given in section 4.2.

Section 2 of this paper collects together some of the elementary theory.

Section 3 then focuses on the most interesting minimal case of  $n = k + 1$ . The known solutions are presented and Smyth's attractive recent treatment of the largest known case ( $n = 10$ ) is discussed. In these minimal cases a solution must have considerable additional structure.

Two related problems are discussed in Section 4. One is due to Erdős and Szekeres the other due to Wright. Both have been open for decades.

Section 6 presents some of the many open problems directly related to these matters.

## 2. ELEMENTARY PROPERTIES

The problem can be stated in three equivalent ways. This is an old result as are most of the results of this section in some form or another. (See for example [7], [11].) In various contexts it is easier to use different forms of the problem.

PROPOSITION 1. *The following are equivalent:*

- (1) 
$$\sum_{i=1}^n \alpha_i^j = \sum_{i=1}^n \beta_i^j \quad \text{for } j = 1, \dots, k$$
- (2) 
$$\deg \left( \prod_{i=1}^n (x - \alpha_i) - \prod_{i=1}^n (x - \beta_i) \right) \leq n - (k + 1)$$
- (3) 
$$(x - 1)^{k+1} \mid \sum_{i=1}^n x^{\alpha_i} - \sum_{i=1}^n x^{\beta_i}.$$

*Proof.* An application of Newton's symmetric polynomial identities shows the equivalence of (1) and (2). To prove the equivalence of (1) and (3) apply  $xd/dx$  to equation (3) and evaluate at one  $k + 1$  times.  $\square$

A solution of the Prouhet-Tarry-Escott problem generates a family of solutions by the following lemma. Any solutions that can be derived from each other in this manner are said to be equivalent.

LEMMA 1. *If  $\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}$  is a solution of degree  $k$ , then so is  $\{M\alpha_1 + K, \dots, M\alpha_n + K\}, \{M\beta_1 + K, \dots, M\beta_n + K\}$  for arbitrary integers  $M, K$ .*

*Proof.* The second form of the problem is clearly preserved when the polynomials  $\prod_{i=1}^n (x - \alpha_i)$  and  $\prod_{i=1}^n (x - \beta_i)$  are scaled and translated by integer constants.  $\square$