Section 4: Concluding remarks

Notice that the convex hull of Y with respect to S contains the ball of radius 4 as measured in the A metric. Furthermore, a simple induction shows that if this convex hull contains the balls of radius n and $n+3$ about the identity (as measured in the metric associated to A) then it contains the ball of radius $n+4$. Thus the convex hull of Y is the whole of $X(\Gamma, S)$. $\square$

SECTION 4: CONCLUDING REMARKS

The type of limit spaces which we considered in Section 2 first arose in work of Morgan and Shalen in which they reinterpreted and generalized Thurston's compactification of Teichmuller space (see [Sha]). The particular topology with respect to which limits are taken in that setting is equivalent to what Paulin has termed "Equivariant Gromov convergence" (see [P1, 2]). It can be shown that the limit tree which we constructed in Section 2 is also a limit in the sense of this topology. We recall Paulin's recent definition:

4.1 DEFINITION. *A sequence of metric spaces Y_n which are equipped with actions by isometries of a fixed group Γ, converge to a metric space Y, which is also equipped with an action of Γ by isometries, if and only if, given any finite set $K \subset Y$, any $\varepsilon > 0$, and any finite subset $P \subset \Gamma$, for sufficiently large n, one can find subsets $K_n \subset Y_n$ and bijections $x_n \mapsto x$ from K_n to K, such that*

$$|d(\gamma x, y) - d_n(\gamma x_n, y_n)| < \varepsilon$$

for all $x, y \in K$ and all $\gamma \in P$.

Limits are not unique in this topology, even if one allows only limit spaces which are complete (cf. [P2], p. 55).

The technique of Equivariant Gromov convergence has been successfully applied in the following settings:

(1) $Y_n = \mathbf{H}^m$ for every integer n and the action of the (abstract) group Γ is discrete and varies with n;

(2) the spaces Y_n are $\mathbf{R}$-trees with isometric Γ-actions;

(3) each Y_n is equal to the Cayley graph of Γ with respect to a fixed set of generators and the action of Γ is left-multiplication twisted by a sequence of homomorphisms $\varphi_n : \Gamma \to \Gamma$.

The situation which we considered in Section 2 belongs to the third of the above cases. In the first two cases, the spaces under consideration enjoy strong convexity properties that allow one to form compact convex hulls of any finite set of points, and under suitable assumptions on the actions being considered one can show that such convex hulls satisfy uniform compactness conditions. The example of the previous section shows that case (3) is less hospitable to analysis of this type. But the arguments which we presented in Section 2 show that this difficulty can be accommodated by using suitable quasi-convex hulls to imitate the more familiar convex hulls used in cases (1) and (2).

If one is interested purely in elucidating the structure of the group Γ under consideration, then arguments using Hausdorff-Gromov convergence, as in Section 2, provide a direct method for constructing a limit object (X_∞, x_∞) which can be used to study properties of Γ. If, on the other hand, one is interested in some kind of representation space for Γ, then it is more useful to formulate results in terms of the above notion of equivariant Gromov convergence. If one is working in a situation where there is an *a priori* different topology to be considered, then one is left with the task of showing that the above notion of equivariant Gromov convergence agrees with the notion of convergence in this other topology. Such verifications have been successfully carried out for various discrete actions of non-elementary hyperbolic groups Γ by Bestvina [B] and Paulin [P1-3]. In particular Paulin has shown that the above notion of convergence leads to the same topology on small actions of a fixed group on $\mathbf{R}$-trees $SLF(\Gamma)$ as the more familiar topology given by length functions.

Somewhat surprisingly, similar Hausdorff-Gromov type arguments do not seem to work so well without some assumption of smallness on the actions considered. For example, the space of all length functions $LF(\Gamma)$ on a fixed group Γ does not seem so amenable to such an analysis. The problem appears to lie with the absence in this generality of any assumption to play the role which the Margulis Lemma plays in the case of discrete actions on $\mathbf{H}^n$. One final remark about length functions: an important subspace of $SLF(\Gamma)$ is $VSLF(\Gamma)$, the space of very small actions introduced by Cohen and Lustig [CL]. Compactness of $VSLF(\Gamma)$ (a result due to Cohen and Lustig) can be proved using equivariant Gromov convergence in the same manner as one shows compactness of $SLF(\Gamma)$; see Paulin [P5].

Rips and Sela ([RS], [S1], [S2]) have made extensive use of variations on the arguments in Section 2 above to study hyperbolic groups. Most strikingly, from Paulin's theorem and deep work of Rips (see [RS], [BF]) one obtains an analogue for hyperbolic groups of the annulus theorem from 3-dimensional topology.

4.2 (Cylindrical Splitting Theorem).

If Γ *is a hyperbolic group with* $Out(\Gamma)$ *infinite, and if* Γ *has one end, then* Γ *is either an HNN-extension or an amalgamated free product over a virtually infinite cyclic group.*

Hyperbolic groups which do not admit a splitting over a virtually cyclic group have been termed *rigid* by Rips and Sela. They show, by a variant of the argument in Section 2 above (termed the Bestvina-Paulin method by Sela) that rigid hyperbolic groups are co-hopf. If Γ is torsion-free and rigid, then they show that there are only finitely many conjugacy classes of embeddings of Γ into any hyperbolic group. Sela [S2] has begun to investigate hopficity for rigid hyperbolic groups. Thus the techniques which we have attempted to exemplify in Section 2 appear to provide an extremely useful tool in the study of hyperbolic groups.

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