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AN ERGODIC ADDING MACHINE ON THE CANTOR SET

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ABSTRACT. We calculate all ergodic measures for a specific function F on the unit interval. The supports of these measures consist of periodic orbits of period 2^n and the classical ternary Cantor set. On the Cantor set, F is topologically conjugate to an "adding machine" in base 2. We show that F is representative of the class of functions with zero topological entropy on the unit interval, already analyzed in the literature, and its behavior is therefore typical of that class.

I. INTRODUCTION

The dynamical behavior of the quadratic function $f_c(x) = x^2 - c$ has been extensively studied as the parameter c is varied. For example, $c_0 = 1.401155189...$ is the smallest value of c for which $f_c(x)$ has infinitely many distinct periodic orbits [1-3]. As c approaches this number through smaller values, the dynamical system, $x \to f_c(x)$, progresses through the famous period doubling route to chaos. When $c = c_0$, the dynamical behavior of $f(x) \equiv f_c(x)$ includes the following properties:

1. There is a Cantor set K which is an attractor and $f : K \to K$
2. All periodic points of f have period 2^n for some n.
3. There are periodic points which are arbitrarily close to K.
4. With the restriction of $f(x)$ to an appropriate interval I such that $f(I) \subset I$, there are just two possibilities for the orbit of a point $x_0 \in I$: either $f^k(x_0)$ is in a periodic orbit for some k, or $f^k(x_0)$ converges to K as k increases.
5. The restriction of f to K is topologically equivalent to a function on 2-adic integers which adds 1 to its argument (this "adding machine" will be described in detail below).

The Cantor set K is sometimes called a Feigenbaum attractor. When $\mu = 3.57...$, the well-known logistic function $g_\mu(x) = \mu x(1 - x)$ exhibits the same dynamical properties [4]. In fact, a large class of dynamical systems exhibiting the properties 1 through 5 has been studied and the ergodic properties analyzed [2, 3, 5].

A particularly simple example of a dynamical system on the interval [0, 1] satisfying 1 through 5 was given and studied by Delahaye [6] and, in a slightly different form, its topological properties (including 1-5) were given in the statements of a series of exercises by Devaney [7]. The function may be described through the concept of the "double" of a function (cf. [7]) as follows: Let $f_0(x) \equiv \frac{1}{3}$ and define $f_n(x)$ recursively by

$$f_{n+1}(x) = \begin{cases} \frac{1}{3} f_n(3x) + \frac{2}{3} & \text{if} \quad x \in \left[0, \frac{1}{3}\right] \\ -\frac{7}{3}x + \frac{14}{9} & \text{if} \quad x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ x - \frac{2}{3} & \text{if} \quad x \in \left[\frac{2}{3}, 1\right] \end{cases}, \tag{1}$$

and

$$F(x) = \lim_{n \to \infty} f_n(x) . \tag{2}$$

It follows that F is continuous on [0, 1] and that it is its own double, i.e.,

$$F(x) = \begin{cases} \frac{1}{3} F(3x) + \frac{2}{3} & \text{if} \quad x \in \left[0, \frac{1}{3}\right] \\ -\frac{7}{3}x + \frac{14}{9} & \text{if} \quad x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ x - \frac{2}{3} & \text{if} \quad x \in \left[\frac{2}{3}, 1\right] \end{cases}. \tag{3}$$

The function F is shown in Figure 1. The notion of the double of a function and its use in studying dynamical systems goes back to Sharkovskii [8]. A general definition of the double of a function, however, will not be needed here.

We will show in the sequel that the function F, like $x^2 - c_0$, is not chaotic. F closely models the behavior of the quadratic function $x^2 - c$ at the critical value c_0 (and many other functions at corresponding critical values of an associated parameter as well) beyond which chaos is present. In addition, the sequence f_n, in its approach to F, exhibits the classical period doubling bifurcations, characteristic of the onset of chaos [7]. The function F is a simple model for understanding the point of transition from nonchaotic behavior to chaotic behavior. In this note we summarize the topological

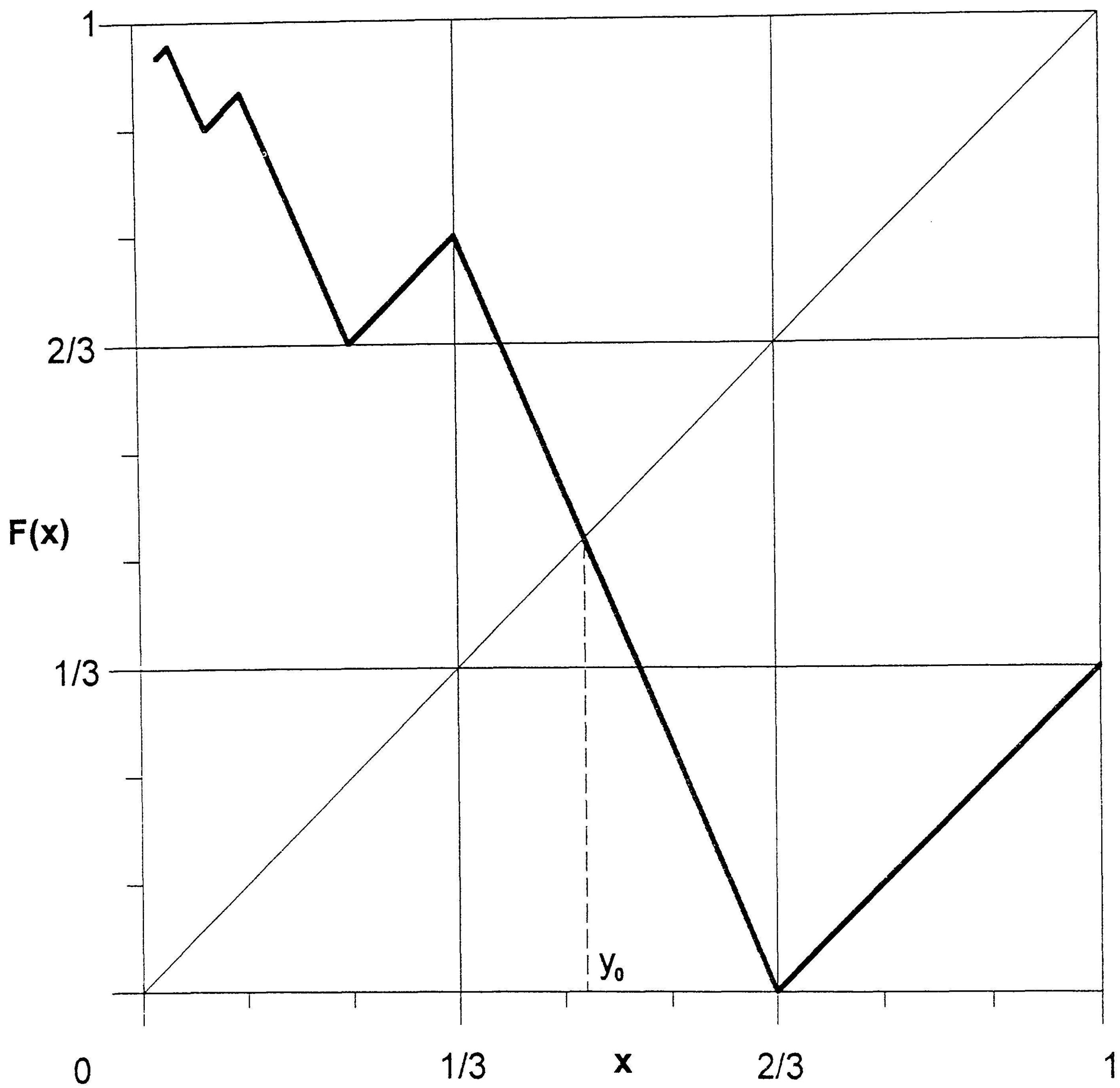

FIGURE 1

Equation (3), the adding machine

properties of the dynamical system $x \to F(x)$ in the form of Theorems 1.1 and 1.2 below and then show how ergodic theory may be used to further analyze the dynamical system. We then indicate how this system may be understood from a more general context developed by Misiurewicz [2, 3] involving topological entropy.

We refer to the following commonly used terms (cf. ref. 7). The point y_0 is a fixed point of F if $F(y_0) = y_0$. The point y is a periodic point of period n if $F^n(y) = y$. The least positive n for which $F^n(y) = y$ is called the *prime period* of y. Hereafter when we refer to a periodic point of period n it shall be understood that n is the prime period. The set of all iterates of a periodic point is a periodic orbit. We shall denote the set of periodic points of period n by $\mathrm{Per}_n(F)$. Finally a point x is *eventually periodic* of period n if x is not periodic but there exists a $j > 0$ such that $F^{n+i}(x) = F^i(x)$ for all $i \geqslant j$. In other words although x is not itself periodic, an iterate of x is.

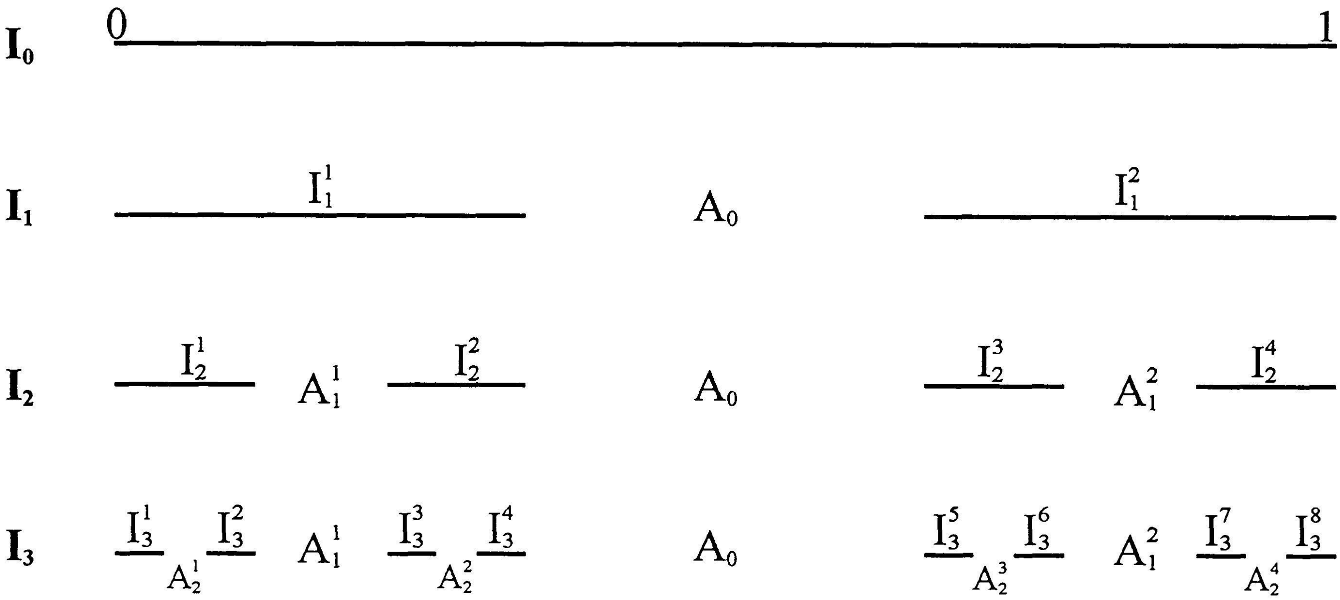

FIGURE 2

Three stages on the way to the Cantor set

Theorem 1.1 below makes reference to the classical ternary Cantor set in $[0, 1]$. We will use the following labeling: The "middle third" intervals that are removed on the way to obtaining the Cantor set are labeled A_n^k, for example, $A_0^1 \equiv A_0 = \left(\frac{1}{3}, \frac{2}{3}\right)$, $A_2^4 = \left(\frac{25}{27}, \frac{26}{27}\right)$, (Figure 2). Set $A_n = \bigcup_{k=1}^{2^n} A_n^k$. Thus A_n consists of 2^n intervals which we number from left to right. We let $(A_{n-1})^c \equiv I_n$, and $I_n = \bigcup_{k=1}^{2^n} I_n^k$, so that I_n also consists of 2^n intervals which we again number from left to right. So, for example, $I_1^1 = \left[0, \frac{1}{3}\right]$, $I_3^8 = \left[\frac{26}{27}, 1\right]$, (Figure 2), but $I_0 = [0, 1]$. Denote the ternary Cantor set by $I_\infty = \bigcap_{n=0}^{\infty} I_n$. It is well-known, and easily deduced, that a real number in $[0, 1]$ is in the Cantor set I_∞ if and only if it has a ternary expansion ("base 3 decimal expansion") of the form $0.\alpha_1\alpha_2\alpha_3\ldots$, where $\alpha_k = 0$ or 2 for each k.

Delahaye's results and Devaney's exercises are slightly extended by Theorem 1.1 below.

THEOREM 1.1. *The function* $F: [0,1] \to [0,1]$ *given by* (2) *satisfies the following properties:*

(a) *For each* n, F *is a cyclic permutation on the collection of intervals* $\{I_n^k : k = 1, \ldots, 2^n\}$, *i.e., for given* k, $F^{2^n}(I_n^k) = I_n^k$, *and for any* $p \neq k$, $p = 1, \ldots, 2^n$, $F^j(I_n^k) = I_n^p$ *for precisely one* j *between* 1 *and* $2^n - 1$.

(b) *For each* $n = 0, 1, 2, \ldots, F$ *has exactly one periodic orbit with period* 2^n *and no other periodic orbits.*

(c) *Every periodic orbit is repelling.* $\mathrm{Per}_{2^n}(F) \subset A_n$ *and* A_n^k *contains exactly one point from* $\mathrm{Per}_{2^n}(F)$ *for each* $k = 1, \ldots, 2^n$ *and each nonnegative integer* n.

(d) *Each point is eventually periodic or converges to* I_∞ *under repeated iterations of* F.

We briefly sketch part of the proof of Theorem 1.1. For $n \geqslant 2$, it can be shown, using induction on n, that

$$F(I_n^k) = I_n^{G(k)},$$

where

$$G(k) = \begin{cases} k - 2^{n-1} & \text{for } 2^{n-1} + 1 \leqslant k \leqslant 2^n \\ 2^n & \text{for } k = 1 \\ k + (2^{N+1} - 3)2^{n-N-1} & \text{for } 2 \leqslant k \leqslant 2^{n-1} \end{cases}$$

and where $N = \left[n - \frac{\log k}{\log 2}\right]$ (and [ ] denotes "integer part"). Part (a) of Theorem 1.1 may now be deduced using this formula and (3).

To check part (b), observe that if $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, then iterates of x by F will eventually move out of $\left[\frac{1}{3}, \frac{2}{3}\right]$ and never return (see Figure 1). If $x \in \left[0, \frac{1}{3}\right]$, then $F(x) \in \left[\frac{2}{3}, 1\right]$, and if $x \in \left[\frac{2}{3}, 1\right]$, then $F(x) \in \left[0, \frac{1}{3}\right]$. Therefore F has no odd periods. An induction argument shows that if $x \in \left[0, \frac{1}{3}\right]$, then $F^{2^n}(x) = \frac{1}{3}F^n(3x)$. To show F does not have any even period orbits other than period 2^n orbits, suppose that there is a period $2^n k$ orbit, where $k > 1$ is an odd number and $n \geqslant 1$. If $x \in \left[0, \frac{1}{3}\right]$ and $F^{2^n k}(x) = x$, then $F^{2^{n-1}k}(3x) = 3x$ and $3x \in \mathrm{Per}_{2^{n-1}k}(F)$. Therefore there is an $x \in \left[0, \frac{1}{3}\right]$ such that $F^{2^{n-1}k}(x) = x$. Continuing in this way we will reach a point such that $F^k(y) = y$, which is impossible since there are no odd period orbits. The existence of a unique orbit of period 2^n follows from (3) and induction on n.

The proofs of parts (c) and (d) use similar ideas and are outlined in the exercises in [7]. $\square$

We turn now to a description of the "adding machine" on the ternary Cantor set and its relationship to F.

A 2-adic integer is an infinite sequence $x = (x_0, x_1, x_2, \ldots)$ where $x_i = 0$ or 1. The collection S of all 2-adic integers is a metric space with the metric $d(x, y) = 2^{-n}$ where $y = (y_0, y_1, y_2, \ldots)$ and n is the smallest integer for which $x_n \neq y_n$. S is the completion of the nonnegative integers with this metric under the identification of the (base 2) integer

$$m = x_0 + x_1 2^1 + x_2 2^2 + \ldots + x_n 2^n$$

with the sequence

$$(x_0, x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) . \tag{4}$$

Define a base 2 addition on S by

$$x + y = z = (z_0, z_1, z_2, \ldots) ,$$

where $z_0 = x_0 + y_0$ if $x_0 + y_0 \leqslant 1$, $z_0 = 0$ if $x_0 + y_0 = 2$ in which case 1 is added to $x_1 + y_1$, which otherwise follows the same rules. The numbers $z_2, z_3, \ldots$ are successively determined in the same manner. Thus, if $x = (x_0, x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$ and $y = (y_0, y_1, y_2, \ldots, y_k, 0, 0, 0, \ldots)$, then $x + y$ corresponds to the usual base 2 arithmetic addition of integers under the identification (4). S is a commutative, compact topological group with this addition. Let us denote the element $(1, 0, 0, \ldots)$ of S by $\mathbf{1}$. Define a map $h : I_\infty \to S$ as follows: If $x \in I_\infty$ has base 3 expansion $0 . \alpha_0 \alpha_1 \alpha_2 \ldots$, where each $\alpha_i = 0$ or 2, then $h(x) = (x_0, x_1, x_2, \ldots)$, where $x_i = 1 - \frac{\alpha_i}{2}$. For example,

$$h(0.02022\ldots) = (1, 0, 1, 0, 0, \ldots) .$$

Theorem 1.2 below was also stated in the same set of exercises in Devaney [7]. We supply the proof for the convenience of the reader.

THEOREM 1.2.

(a) *The function h is a homeomorphism from the ternary Cantor set I_∞ to the 2-adic integers S.*

(b) *$F(I_\infty) = I_\infty$, and F restricted to I_∞ is topologically conjugate by h to the addition of $\mathbf{1}$ on 2-adic integers, i.e., $h(F(x)) = h(x) + \mathbf{1}$.*

(c) *The F-orbit of each point in I_∞ is dense in I_∞.*

Proof. (a) h is clearly one-to-one and onto. To see that h^{-1} is continuous, let $\varepsilon > 0$ be given and choose n so that $3^{-n-1} < \varepsilon$ and let $\delta = 2^{-n}$. If $x, y \in S$ and $d(x, y) \leqslant \delta$, then

$$| h^{-1}(x) - h^{-1}(y) | = | 0.000 \ldots 0 \alpha_n \alpha_{n+1} \ldots | \tag{5}$$

where the first $n-1$ digits on the right side of (5) are zeros and the number on the right is expressed in base 3 so that $\alpha_k = 0$ or 2. Consequently $|h^{-1}(x) - h^{-1}(y)| \leqslant 3^{-n-1} < \varepsilon$. Since h^{-1} is a continuous bijection and I_∞ is compact, it follows from a well-known theorem in topology that h is continuous and therefore h is a homeomorphism.

(b) Suppose $x \in I_\infty \cap \left[\frac{2}{3}, 1\right]$ and let the base 3 expansion of x be given by $x = 0.2\alpha_1\alpha_2\alpha_3\ldots$, thus $h(x) = (0, x_1, x_2, \ldots)$, where $x_i = 1 - \frac{\alpha_i}{2}$. Then $F(x) = x - \frac{2}{3}$ has base 3 expansion given by $F(x) = 0.0\alpha_1\alpha_2\alpha_3\ldots$. Therefore $h(F(x)) = (1, x_1, x_2, \ldots)$ which is the same as $h(x) + \mathbf{1}$. If $x = 0$, then $h(F(0)) = (0, 0, 0, \ldots) = h(0) + \mathbf{1}$. If $x \in I_\infty \cap \left(0, \frac{1}{3}\right]$, then $x \in \left[\frac{2}{3^i}, \frac{1}{3^{i-1}}\right)$ for some $i \geqslant 2$. A brief calculation shows that

$$F(x) = x + 0.\overbrace{2\ldots2}^{i-2}1\overset{\overset{i}{\downarrow}}{1}\bar{0} \tag{6}$$

where the number on the right is expressed in base 3 and the second "1" occurs i places after the decimal point. It now follows from (6) and base 3 addition that $h(F(x)) = h(x) + \mathbf{1}$ and therefore $F(I_\infty) = I_\infty$.

(c) Since S is the completion of the nonnegative integers under the identification (4), the set

$$\{(x_0, x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) : n = 0, 1, 2, \ldots, x_i = 0 \text{ or } 1\}$$

which equals $\{n\mathbf{1} : n = 0, 1, 2, \ldots\}$ is dense in S. It is easy to establish that the map which takes x to $x + z$ is a homeomorphism from S to S for any fixed $z \in S$. Thus

$$\{n\mathbf{1} : n = 0, 1, 2, \ldots\} + z$$

is dense in S for any z. But $h^{-1}(\{n\mathbf{1} : n = 0, 1, 2, \ldots\} + z)$ is precisely the F-orbit of $h^{-1}(z)$. Therefore the F-orbit of any $y = h^{-1}(z) \in I_\infty$ is dense in I_∞. □

II. Ergodic measures for F

A measure μ on a set X is called a probability measure if $\mu(X) = 1$; the pair (X, μ) is then called a probability space. Given a measurable transformation $T: X \to X$ on a probability space (X, μ), μ is T-invariant if $\mu = \mu \circ T^{-1}$, i.e., for any measurable set $B \subset X$, $\mu(B) = \mu(T^{-1}(B))$. The probability measure μ is ergodic if $T^{-1}(A) = A$ implies that $\mu(A)$ is 0 or 1.