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## ELLIPTIC SPACES II

by Yves FELIX, Stephen HALPERIN<sup>1)</sup> and Jean-Claude THOMAS<sup>2)</sup>

**ABSTRACT.** A simply connected finite *CW* complex  $X$  is *elliptic* if the homology of its loop space (coefficients in any field) grows at most polynomially. We show that in all other cases the loop space homology grows at least semi-exponentially, and we exhibit a number of geometrically interesting classes of spaces as elliptic, including:  $H$  spaces, homogeneous spaces, Poincaré duality complexes whose mod  $p$  cohomology is doubly generated (any  $p$ ) and Dupin hypersurfaces in  $S^{n+1}$ .

### 1. INTRODUCTION

Let  $X$  be a simply connected finite *CW* complex, with loop space  $\Omega X$ , and denote by  $\mathbf{F}_p$ , the prime field of characteristic  $p$ ,  $p$  prime or zero. Our first main result asserts a dichotomy for the size of the loop space homology  $H_*(\Omega X; \mathbf{F}_p)$ :

**THEOREM A.** *Let  $X$  be a simply connected finite *CW* complex. For each  $p$  (prime or zero) there are exactly two possibilities: either*

(i) *There are constants  $C > 0$  and  $r \in \mathbf{N}$  such that*

$$\sum_{i=0}^n \dim H_i(\Omega X; \mathbf{F}_p) \leq Cn^r, \quad n \geq 1,$$

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*Key words:* loop space homology, depth, polynomial growth, Poincaré complex, elliptic, Dupin hypersurface.

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or else

(ii) There are constants  $K > 1$  and  $N \in \mathbf{N}$  such that

$$\sum_{i=0}^n \dim H_i(\Omega X; \mathbf{F}_p) \geq K\sqrt{n}, \quad n \geq N.$$

In case (i) the loop space homology grows *at most polynomially*, and  $X$  is  $\mathbf{Z}_{(p)}$ -elliptic in the sense of [6]. If (i) holds for all  $p$  then  $X$  is elliptic. The main theorems of [6] assert that if  $X$  is elliptic then  $X$  is a Poincaré complex and that  $H_*(\Omega X; \mathbf{Z})$  is a finitely generated left noetherian ring.

In case (ii) above the loop space homology grows *at least semi-exponentially*. However, when  $p = 0$  [2] or  $p \geq \dim X$  [8], it can be shown that even the primitive subspace of  $H_*(\Omega X; \mathbf{F}_p)$  grows exponentially (implying the same result for  $H_*(\Omega X; \mathbf{F}_p)$ ), and we conjecture that this should hold true for all  $p$ .

In the dichotomy of Theorem A, the generic situation is (ii): elliptic spaces are rare within the class of all simply connected finite  $CW$  complexes. However a number of geometrically interesting spaces are elliptic, and our second objective in this note is to show that these include the following classes of spaces (provided they are simply connected):

finite  $H$ -spaces,

homogeneous spaces,

spaces admitting a fibration  $F \rightarrow X \rightarrow B$  with  $F, B$  elliptic,

Poincaré complexes  $X$  such that for each  $p$ , the algebra  $H^*(X; \mathbf{F}_p)$  is generated by two elements,

Dupin hypersurfaces in  $S^{n+1}$ ,

closed manifolds admitting a smooth action by a compact Lie group, with a simply connected codimension one orbit,

connected sums  $M \# N$  with the algebras  $H^*(M; \mathbf{Z})$  and  $H^*(N; \mathbf{Z})$  each generated by a single class.

This note is sequel to “Elliptic Spaces” [6]. In particular, it supersedes the preprint “Dupin hypersurfaces are elliptic” referred to in [6].

## 2. THE DICHOTOMY

Consider first any simply connected space  $X$  with each  $H_i(X; \mathbf{F}_p)$  finite dimensional. Then  $G = H_*(\Omega X; \mathbf{F}_p)$  is a graded cocommutative Hopf algebra satisfying  $G_0 = \mathbf{F}_p$  and each  $G_i$  is finite dimensional. The *depth* of  $G$

is the least integer  $m$  such that  $\text{Ext}_G^m(\mathbf{F}_p; G) \neq 0$ ; if  $\text{Ext}_G(\mathbf{F}_p; G) \equiv 0$  we say  $G$  has *infinite depth*. In [3: Theorem A] it is shown that

$$\text{depth } H_*(\Omega X; \mathbf{F}_p) \leq LS \text{ cat } X .$$

Thus the depth is finite when  $X$  has the weak homotopy type of a finite CW complex.

On the other hand suppose  $G$  is any graded cocommutative Hopf algebra with  $G_0 = \mathbf{F}_p$  and each  $G_i$  finite dimensional. We call  $G$  *elliptic* [7] if  $G$  is a finitely generated nilpotent Hopf algebra. According to [4; Theorem A] this is equivalent to the condition:

$$\text{depth } G < \infty \quad \text{and} \quad \sum_{i=0}^n \dim G_i \leq Cn^r \text{ (fixed } C, r, \text{ all } n) .$$

In view of these remarks, Theorem A follows from

**THEOREM 2.1.** *Let  $G$  be a cocommutative Hopf algebra of finite depth such that  $G_0 = \mathbf{F}_p$  and each  $G_i$  is finite dimensional. Then there are exactly two possibilities:*

(1)  $G$  is elliptic, and for some  $r \in \mathbf{N}$  there are positive constants  $C_1, C_2$  such that

$$C_1 n^r \leq \sum_{i=0}^n \dim G_i \leq C_2 n^r, \quad n \geq 1 ;$$

(2) For some constants  $K > 1, N \in \mathbf{N}$

$$\sum_{i=0}^n \dim G_i \geq K\sqrt[n]{n}, \quad n \geq N .$$

*Proof.* Consider the formal power series  $G(z) = \sum_{i=0}^{\infty} \dim G_i z^i$ , and for

two formal power series  $f = \sum_{i=0}^{\infty} a_i z^i$  and  $g = \sum_{i=0}^{\infty} b_i z^i$  write  $f \leq_c g$  if

$$(2.1) \quad \sum_{i=0}^n a_i \leq \sum_{i=0}^n b_i, \quad \text{all } n .$$

We shall first show that there are exactly two possibilities:

(2.2) For some  $r \in \mathbf{N}$  there are positive constants  $C_1, C_2$  such that

$$C_1 n^r \leq \sum_{i=0}^n \dim G_i \leq C_2 n^r, \quad n \geq 1 ;$$

(2.3) For some  $k \in \mathbb{N}$ .

$$G(z) \underset{c}{\gg} \prod_{i=1}^{\infty} [1 + (z^k)^i] .$$

Indeed, suppose  $\sum_{i=0}^n \dim G_i \leq C_2 n^r$  for all  $n$ , some  $C_2$  and  $r$ . Then by [4; Theorem B],  $G$  is elliptic and hence [7; Prop. 3.6] the formal power series  $G(z)$  has the form

$$G(z) = \frac{\prod_{j=1}^s (1 + z^{k_j} + \cdots + z^{(n_j-1)k_j})}{\prod_{i=1}^r (1 - z^{l_i})} .$$

It follows at once that (2.2) is satisfied.

Conversely, we assume there is no  $C, r$  for which  $\sum_{i=0}^n \dim G_i \leq Cn^r$ , all  $n$ , and prove (2.3). Let  $x_1, x_2, \dots$  be a sequence of generators of the algebra  $G$  with  $\deg x_1 \leq \deg x_2 \leq \cdots$ . The subalgebra  $G(i)$  generated by  $x_1, \dots, x_i$  is then a sub Hopf algebra. Now according to [4; Prop. 3.1] there is some  $q$  such that  $G(i)$  has finite depth,  $i \geq q$ . Moreover by [7; Prop. 3.5]  $G(l)$  is not elliptic for some  $l \geq q$ . Set  $H = G(l)$ ; it is a finitely generated non-elliptic Hopf algebra of finite depth, and  $\dim G_i \geq \dim H_i$ .

Next, let  $R$  be the sum of the solvable normal sub Hopf algebras of  $H$ . Then [3; Theorem C]  $R$  is elliptic. Hence [7; Prop. 3.1] and [3; Prop. 3.1] the quotient Hopf algebra  $H // R$  has finite depth, but [7; Prop. 3.3]  $H // R$  is not elliptic. Clearly, however,  $H // R$  is finitely generated and has no central primitive elements. Now by [4; Prop. 3] there is an integer  $n_0$  and an infinite sequence of non zero primitive elements  $y_i \in H // R$  such that for all  $i$ ,  $\deg y_i \leq \deg y_{i+1} \leq \deg y_i + n_0$ . A linear embedding

$$\bigotimes_{i=1}^{\infty} \mathbb{F}_p[y_i] / y_i^2 \rightarrow H // R$$

is then defined by  $y_1^{\varepsilon_1} \otimes \cdots \otimes y_m^{\varepsilon_m} \rightarrow y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m}$ , and so

$$\prod_{i=1}^{\infty} (1 + z^{\deg y_i}) \underset{c}{\ll} (H // R)(z) \underset{c}{\ll} H(z) \underset{c}{\ll} G(z) .$$

Since  $\deg y_{i+1} \leq in_0 + \deg y_1$  it is sufficient to take  $k = \max(\deg y_1, n_0)$  to achieve (2.3).

It remains to deduce the inequality (2) from (2.3). If the inequality (2) holds for some power series  $h(z)$  it will also hold for  $h(z^k)$ , at the cost of replacing  $K$  by  $K^{\frac{1}{2k}}$ . By (2.3) we are thus reduced to showing that the power series

$$\sum_{i=0}^{\infty} q_i z^i = \prod_{i=0}^{\infty} (1 + z^i)$$

satisfies (2). But this is an immediate consequence of a theorem of Hardy and Ramanujan [10].  $\square$

COROLLARY OF PROOF. *If  $G$  satisfies the hypotheses of Theorem 2.1 (2) then for some  $k \in \mathbb{N}$ ,*

$$G(z) \underset{c}{\geq} \prod_{i=1}^{\infty} [1 + (z^k)^i]. \quad \square$$

### 3. ELLIPTIC SPACES

In this section we establish the ellipticity of the spaces listed in the introduction.

#### 3.1. Finite simply connected $H$ -spaces, $X$ .

Because  $X$  is an  $H$ -space,  $H_*(\Omega X; \mathbb{F}_p)$  is commutative, all  $p$ . Since it has finite depth [3; Theorem A] it is elliptic [7; Prop. 3.2]. Hence  $X$  is elliptic.

#### 3.2. Simply connected homogeneous spaces, $G // H$ .

We may suppose that  $G$  is simply connected, and hence elliptic by §3. The fibration  $G \rightarrow G/H \rightarrow BH$  loops to the fibration  $\Omega G \rightarrow \Omega(G/H) \rightarrow H$  in which  $\pi_1(H)$  acts trivially in  $H_*(\Omega G; \mathbb{F}_p)$  [1; Lemma 5.1]. Thus we can use the Serre spectral sequence to deduce polynomial growth for  $H_*(\Omega(G/H); \mathbb{F}_p)$  from the same property for  $H_*(\Omega G; \mathbb{F}_p)$ .

#### 3.3. Fibrations $F \rightarrow X \rightarrow B$ with $F, B$ elliptic.

Here all spaces are simply connected and we can apply the Serre spectral sequence to deduce that  $H_*(X; \mathbb{Z})$  is concentrated in finitely many degrees, and finitely generated in each. Hence  $X$  has the weak homotopy type of a finite  $CW$  complex. Loop the fibration  $F \rightarrow X \rightarrow B$  and use the fact that  $H_*(\Omega F; \mathbb{F}_p)$  and  $H_*(\Omega B; \mathbb{F}_p)$  grow polynomially to deduce the same property for  $H_*(\Omega X; \mathbb{F}_p)$ .

### 3.4. Simply connected Poincaré complexes $X$ with $H^*(X; \mathbb{F}_p)$ at most doubly generated.

Suppose  $p \neq 2$  and  $H = H^*(X; \mathbb{F}_p)$  contains an element of odd degree. Then it has an odd generator  $\alpha$ . Using Poincaré duality it is easy to see that there are only three possibilities for the algebra  $H$ :

$$H = \Lambda \alpha \quad \text{or} \quad \Lambda \alpha \otimes \Lambda \beta \quad \text{or} \quad \Lambda \alpha \otimes \mathbb{F}_p[\beta] / \beta^k.$$

In each case a simple, classical computation [11] produces  $\text{Tor}^H(\mathbb{F}_p, \mathbb{F}_p)$  and shows that it grows polynomially. Since the Eilenberg-Moore spectral sequence converges from  $\text{Tor}^H(\mathbb{F}_p, \mathbb{F}_p)$  to  $H^*(\Omega X; \mathbb{F}_p)$ ,  $H^*(\Omega X; \mathbb{F}_p)$  also has this property.

In all other cases ( $p = 2$  or  $H$  concentrated in even degrees)  $H$  is a commutative local ring in the classic sense. Because  $H$  satisfies Poincaré duality it is a Gorenstein ring. Now a theorem of Wiebe [12; Korollar p. 268] asserts (because  $H$  has at most two generators) that  $H$  is a polynomial algebra divided by a regular sequence. It is thus easy (and classical [11]) to compute  $\text{Tor}^H(\mathbb{F}_p, \mathbb{F}_p)$ , and deduce that it grows polynomially. Hence so does  $H_*(\Omega X; \mathbb{F}_p)$ .

### 3.5. Simply connected Dupin hypersurfaces $E$ in $S^{n+1}$ .

In [9; Table 2.1] are listed the possibilities for  $H_*(E; \mathbb{Z})$ . We divide these into three cases, using the notation of [9].

*Case (a):*  $E$  has the same integral homology as  $S^k$  or as  $S^k \times S^l$ .

In this case Poincaré duality shows that  $E$  has the same integral cohomology ring as  $S^k$  or as  $S^k \times S^l$ , and we can apply 3.4.

*Case (b):*  $E$  has the rational homotopy type of  $A_3(2)$ ,  $A_3(4)$ ,  $A_3(8)$ ,  $A_4(2)$  or  $A_6(2)$ .

In these cases the calculations of [9; §6] show explicitly that the ring  $H^*(E; \mathbb{Z})$  is torsion free and generated by two elements. Thus each  $H^*(E; \mathbb{F}_p)$  is doubly generated, and we can apply Wiebe's result as in 3.4.

*Case (c):*  $E$  has the integral homology of  $S^k \times S^l \times S^{k+l}$ , with  $k < l$ .

We need, in this case, to recall from [9; §2] that there are linear sphere bundles

$$S^k \rightarrow E \xrightarrow{\pi_0} B \quad \text{and} \quad S^l \rightarrow E \xrightarrow{\pi_1} B_1$$

with  $B_0, B_1$  simply connected focal submanifolds of  $S^{n+1}$ . Moreover if  $D_0, D_1$  denote the corresponding disk bundles with boundary  $E$  then  $S^{n+1} = D_0 \cup_E D_1$ .

Fix  $p \geq 0$  and consider the Serre spectral sequence for the fibration  $S^k \rightarrow E \rightarrow B_0$  with coefficients in  $\mathbb{F}_p$ . If this fails to collapse then  $H^k(\pi_0): H^k(B_0; \mathbb{F}_p) \rightarrow H^k(E; \mathbb{F}_p)$  is surjective. Since  $l > k$  it is always true that  $H^k(\pi_1)$  is surjective. Choose classes  $\alpha \in H^k(B_0; \mathbb{F}_p)$ ,  $\beta \in H^k(B_1; \mathbb{F}_p)$  mapping to the same non-zero class in  $H^k(E; \mathbb{F}_p)$ . The Mayer-Vietoris sequence for the decomposition  $S^{n+1} = D_0 \cup_E D_1$  then gives a class  $\gamma \in H^k(S^{n+1}; \mathbb{F}_p)$  restricting to  $\alpha$  and  $\beta$ , which is absurd.

Thus the spectral sequence for  $S^k \rightarrow E \rightarrow B_0$  collapses and so  $H_*(B_0; \mathbb{F}_p) \cong H_*(S^l \times S^{l+k}; \mathbb{F}_p)$ . Using Poincaré duality for  $B_0$  we see that  $H^*(B_0; \mathbb{F}_p)$  and  $H^*(S^l \times S^{l+k}; \mathbb{F}_p)$  are isomorphic as graded algebras. Thus  $B_0$  is elliptic by 3.4 and  $E$  is elliptic by 3.3.

*3.6. Simply connected closed manifolds  $M$  with a smooth action by a compact Lie group  $G$ , having a simply connected codimension one orbit.*

Here we may assume  $G$  is connected. Let the orbit be  $G/K$ , and convert the inclusion of  $G/K$  into a fibration  $F \rightarrow G/K \rightarrow M$ . From [9; Table 1.5] we see that for any  $p$ ,  $\dim H_i(F; \mathbb{F}_p) \leq 2$ , all  $i$ . Thus applying the Serre spectral sequence to the fibration  $\Omega(G/K) \rightarrow \Omega M \rightarrow F$  and using 3.1 for  $G/K$  we see that  $H_*(\Omega M; \mathbb{F}_p)$  grows polynomially.

*3.7. Simply connected manifolds  $M \# N$  with each of the rings  $H^*(M; \mathbb{Z})$ ,  $H^*(N; \mathbb{Z})$  generated by a single class.*

By Van Kampen's theorem both  $M$  and  $N$  are simply connected, and so their fundamental cohomology classes are not torsion. Since each ring is monogenic,  $H^*(M; \mathbb{Z})$  and  $H^*(N; \mathbb{Z})$  are torsion free. Thus  $H^*(M; \mathbb{F}_p)$  and  $H^*(N; \mathbb{F}_p)$  are also monogenic, and so  $H^*(M \# N; \mathbb{F}_p)$  is doubly generated. Now apply 3.4.

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