

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ELLIPTIC SPACES II
Autor: Felix, Yves / Halperin, Stephen / Thomas, Jean-Claude
DOI: <https://doi.org/10.5169/seals-60412>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ELLIPTIC SPACES II

by Yves FELIX, Stephen HALPERIN¹⁾ and Jean-Claude THOMAS²⁾

ABSTRACT. A simply connected finite CW complex X is *elliptic* if the homology of its loop space (coefficients in any field) grows at most polynomially. We show that in all other cases the loop space homology grows at least semi-exponentially, and we exhibit a number of geometrically interesting classes of spaces as elliptic, including: H spaces, homogeneous spaces, Poincaré duality complexes whose mod p cohomology is doubly generated (any p) and Dupin hypersurfaces in S^{n+1} .

1. INTRODUCTION

Let X be a simply connected finite CW complex, with loop space ΩX , and denote by \mathbf{F}_p , the prime field of characteristic p , p prime or zero. Our first main result asserts a dichotomy for the size of the loop space homology $H_*(\Omega X; \mathbf{F}_p)$:

THEOREM A. *Let X be a simply connected finite CW complex. For each p (prime or zero) there are exactly two possibilities: either*

(i) *There are constants $C > 0$ and $r \in \mathbf{N}$ such that*

$$\sum_{i=0}^n \dim H_i(\Omega X; \mathbf{F}_p) \leq Cn^r, \quad n \geq 1,$$

Key words: loop space homology, depth, polynomial growth, Poincaré complex, elliptic, Dupin hypersurface.

AMS Mathematical subject classification: 55P35, 57P10, 57T25, 57S25, 53C25.

Research partially supported by a NATO travel grant held by the three authors.

¹⁾ Research partially supported by an NSERC operating grant.

²⁾ URA-D751 au CNRS.

or else

(ii) There are constants $K > 1$ and $N \in \mathbf{N}$ such that

$$\sum_{i=0}^n \dim H_i(\Omega X; \mathbf{F}_p) \geq K\sqrt{n}, \quad n \geq N.$$

In case (i) the loop space homology grows *at most polynomially*, and X is $\mathbf{Z}_{(p)}$ -elliptic in the sense of [6]. If (i) holds for all p then X is elliptic. The main theorems of [6] assert that if X is elliptic then X is a Poincaré complex and that $H_*(\Omega X; \mathbf{Z})$ is a finitely generated left noetherian ring.

In case (ii) above the loop space homology grows *at least semi-exponentially*. However, when $p = 0$ [2] or $p \geq \dim X$ [8], it can be shown that even the primitive subspace of $H_*(\Omega X; \mathbf{F}_p)$ grows exponentially (implying the same result for $H_*(\Omega X; \mathbf{F}_p)$), and we conjecture that this should hold true for all p .

In the dichotomy of Theorem A, the generic situation is (ii): elliptic spaces are rare within the class of all simply connected finite CW complexes. However a number of geometrically interesting spaces are elliptic, and our second objective in this note is to show that these include the following classes of spaces (provided they are simply connected):

finite H -spaces,

homogeneous spaces,

spaces admitting a fibration $F \rightarrow X \rightarrow B$ with F, B elliptic,

Poincaré complexes X such that for each p , the algebra $H^*(X; \mathbf{F}_p)$ is generated by two elements,

Dupin hypersurfaces in S^{n+1} ,

closed manifolds admitting a smooth action by a compact Lie group, with a simply connected codimension one orbit,

connected sums $M \# N$ with the algebras $H^*(M; \mathbf{Z})$ and $H^*(N; \mathbf{Z})$ each generated by a single class.

This note is sequel to ‘‘Elliptic Spaces’’ [6]. In particular, it supersedes the preprint ‘‘Dupin hypersurfaces are elliptic’’ referred to in [6].

2. THE DICHOTOMY

Consider first any simply connected space X with each $H_i(X; \mathbf{F}_p)$ finite dimensional. Then $G = H_*(\Omega X; \mathbf{F}_p)$ is a graded cocommutative Hopf algebra satisfying $G_0 = \mathbf{F}_p$ and each G_i is finite dimensional. The *depth* of G

is the least integer m such that $\text{Ext}_G^m(\mathbf{F}_p; G) \neq 0$; if $\text{Ext}_G(\mathbf{F}_p; G) \equiv 0$ we say G has *infinite depth*. In [3: Theorem A] it is shown that

$$\text{depth } H_*(\Omega X; \mathbf{F}_p) \leq LS \text{ cat } X .$$

Thus the depth is finite when X has the weak homotopy type of a finite CW complex.

On the other hand suppose G is any graded cocommutative Hopf algebra with $G_0 = \mathbf{F}_p$ and each G_i finite dimensional. We call G *elliptic* [7] if G is a finitely generated nilpotent Hopf algebra. According to [4; Theorem A] this is equivalent to the condition:

$$\text{depth } G < \infty \quad \text{and} \quad \sum_{i=0}^n \dim G_i \leq Cn^r \text{ (fixed } C, r, \text{ all } n) .$$

In view of these remarks, Theorem A follows from

THEOREM 2.1. *Let G be a cocommutative Hopf algebra of finite depth such that $G_0 = \mathbf{F}_p$ and each G_i is finite dimensional. Then there are exactly two possibilities:*

(1) G is elliptic, and for some $r \in \mathbf{N}$ there are positive constants C_1, C_2 such that

$$C_1 n^r \leq \sum_{i=0}^n \dim G_i \leq C_2 n^r, \quad n \geq 1 ;$$

(2) For some constants $K > 1, N \in \mathbf{N}$

$$\sum_{i=0}^n \dim G_i \geq K\sqrt[n]{n}, \quad n \geq N .$$

Proof. Consider the formal power series $G(z) = \sum_{i=0}^{\infty} \dim G_i z^i$, and for

two formal power series $f = \sum_{i=0}^{\infty} a_i z^i$ and $g = \sum_{i=0}^{\infty} b_i z^i$ write $f \leq_c g$ if

$$(2.1) \quad \sum_{i=0}^n a_i \leq \sum_{i=0}^n b_i, \quad \text{all } n .$$

We shall first show that there are exactly two possibilities:

(2.2) For some $r \in \mathbf{N}$ there are positive constants C_1, C_2 such that

$$C_1 n^r \leq \sum_{i=0}^n \dim G_i \leq C_2 n^r, \quad n \geq 1 ;$$

(2.3) For some $k \in \mathbf{N}$.

$$G(z) \underset{c}{\gg} \prod_{i=1}^{\infty} [1 + (z^k)^i].$$

Indeed, suppose $\sum_{i=0}^n \dim G_i \leq C_2 n^r$ for all n , some C_2 and r . Then by [4; Theorem B], G is elliptic and hence [7; Prop. 3.6] the formal power series $G(z)$ has the form

$$G(z) = \frac{\prod_{j=1}^s (1 + z^{k_j} + \cdots + z^{(n_j-1)k_j})}{\prod_{i=1}^r (1 - z^{l_i})}.$$

It follows at once that (2.2) is satisfied.

Conversely, we assume there is no C, r for which $\sum_{i=0}^n \dim G_i \leq Cn^r$, all n , and prove (2.3). Let x_1, x_2, \dots be a sequence of generators of the algebra G with $\deg x_1 \leq \deg x_2 \leq \cdots$. The subalgebra $G(i)$ generated by x_1, \dots, x_i is then a sub Hopf algebra. Now according to [4; Prop. 3.1] there is some q such that $G(i)$ has finite depth, $i \geq q$. Moreover by [7; Prop. 3.5] $G(l)$ is not elliptic for some $l \geq q$. Set $H = G(l)$; it is a finitely generated non-elliptic Hopf algebra of finite depth, and $\dim G_i \geq \dim H_i$.

Next, let R be the sum of the solvable normal sub Hopf algebras of H . Then [3; Theorem C] R is elliptic. Hence [7; Prop. 3.1] and [3; Prop. 3.1] the quotient Hopf algebra $H // R$ has finite depth, but [7; Prop. 3.3] $H // R$ is not elliptic. Clearly, however, $H // R$ is finitely generated and has no central primitive elements. Now by [4; Prop. 3] there is an integer n_0 and an infinite sequence of non zero primitive elements $y_i \in H // R$ such that for all i , $\deg y_i \leq \deg y_{i+1} \leq \deg y_i + n_0$. A linear embedding

$$\bigotimes_{i=1}^{\infty} \mathbf{F}_p[y_i]/y_i^2 \rightarrow H // R$$

is then defined by $y_1^{\varepsilon_1} \otimes \cdots \otimes y_m^{\varepsilon_m} \rightarrow y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m}$, and so

$$\prod_{i=1}^{\infty} (1 + z^{\deg y_i}) \underset{c}{\ll} (H // R)(z) \underset{c}{\ll} H(z) \underset{c}{\ll} G(z).$$

Since $\deg y_{i+1} \leq in_0 + \deg y_1$ it is sufficient to take $k = \max(\deg y_1, n_0)$ to achieve (2.3).

It remains to deduce the inequality (2) from (2.3). If the inequality (2) holds for some power series $h(z)$ it will also hold for $h(z^k)$, at the cost of replacing K by $K^{\frac{1}{2k}}$. By (2.3) we are thus reduced to showing that the power series

$$\sum_{i=0}^{\infty} q_i z^i = \prod_{i=0}^{\infty} (1 + z^i)$$

satisfies (2). But this is an immediate consequence of a theorem of Hardy and Ramanujan [10]. \square

COROLLARY OF PROOF. *If G satisfies the hypotheses of Theorem 2.1 (2) then for some $k \in \mathbf{N}$,*

$$G(z) \underset{c}{\geq} \prod_{i=1}^{\infty} [1 + (z^k)^i]. \quad \square$$

3. ELLIPTIC SPACES

In this section we establish the ellipticity of the spaces listed in the introduction.

3.1. Finite simply connected H -spaces, X .

Because X is an H -space, $H_*(\Omega X; \mathbf{F}_p)$ is commutative, all p . Since it has finite depth [3; Theorem A] it is elliptic [7; Prop. 3.2]. Hence X is elliptic.

3.2. Simply connected homogeneous spaces, $G // H$.

We may suppose that G is simply connected, and hence elliptic by §3. The fibration $G \rightarrow G/H \rightarrow BH$ loops to the fibration $\Omega G \rightarrow \Omega(G/H) \rightarrow H$ in which $\pi_1(H)$ acts trivially in $H_*(\Omega G; \mathbf{F}_p)$ [1; Lemma 5.1]. Thus we can use the Serre spectral sequence to deduce polynomial growth for $H_*(\Omega(G/H); \mathbf{F}_p)$ from the same property for $H_*(\Omega G; \mathbf{F}_p)$.

3.3. Fibrations $F \rightarrow X \rightarrow B$ with F, B elliptic.

Here all spaces are simply connected and we can apply the Serre spectral sequence to deduce that $H_*(X; \mathbf{Z})$ is concentrated in finitely many degrees, and finitely generated in each. Hence X has the weak homotopy type of a finite CW complex. Loop the fibration $F \rightarrow X \rightarrow B$ and use the fact that $H_*(\Omega F; \mathbf{F}_p)$ and $H_*(\Omega B; \mathbf{F}_p)$ grow polynomially to deduce the same property for $H_*(\Omega X; \mathbf{F}_p)$.

3.4. *Simply connected Poincaré complexes X with $H^*(X; \mathbf{F}_p)$ at most doubly generated.*

Suppose $p \neq 2$ and $H = H^*(X; \mathbf{F}_p)$ contains an element of odd degree. Then it has an odd generator α . Using Poincaré duality it is easy to see that there are only three possibilities for the algebra H :

$$H = \Lambda\alpha \quad \text{or} \quad \Lambda\alpha \otimes \Lambda\beta \quad \text{or} \quad \Lambda\alpha \otimes \mathbf{F}_p[\beta]/\beta^k.$$

In each case a simple, classical computation [11] produces $\text{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ and shows that it grows polynomially. Since the Eilenberg-Moore spectral sequence converges from $\text{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ to $H^*(\Omega X; \mathbf{F}_p)$, $H^*(\Omega X; \mathbf{F}_p)$ also has this property.

In all other cases ($p = 2$ or H concentrated in even degrees) H is a commutative local ring in the classic sense. Because H satisfies Poincaré duality it is a Gorenstein ring. Now a theorem of Wiebe [12; Korollar p. 268] asserts (because H has at most two generators) that H is a polynomial algebra divided by a regular sequence. It is thus easy (and classical [11]) to compute $\text{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$, and deduce that it grows polynomially. Hence so does $H_*(\Omega X; \mathbf{F}_p)$.

3.5. *Simply connected Dupin hypersurfaces E in S^{n+1} .*

In [9; Table 2.1] are listed the possibilities for $H_*(E; \mathbf{Z})$. We divide these into three cases, using the notation of [9].

Case (a): E has the same integral homology as S^k or as $S^k \times S^l$.

In this case Poincaré duality shows that E has the same integral cohomology ring as S^k or as $S^k \times S^l$, and we can apply 3.4.

Case (b): E has the rational homotopy type of $A_3(2)$, $A_3(4)$, $A_3(8)$, $A_4(2)$ or $A_6(2)$.

In these cases the calculations of [9; §6] show explicitly that the ring $H^*(E; \mathbf{Z})$ is torsion free and generated by two elements. Thus each $H^*(E; \mathbf{F}_p)$ is doubly generated, and we can apply Wiebe's result as in 3.4.

Case (c): E has the integral homology of $S^k \times S^l \times S^{k+l}$, with $k < l$.

We need, in this case, to recall from [9; §2] that there are linear sphere bundles

$$S^k \rightarrow E \xrightarrow{\pi_0} B \quad \text{and} \quad S^l \rightarrow E \xrightarrow{\pi_1} B_1$$

with B_0, B_1 simply connected focal submanifolds of S^{n+1} . Moreover if D_0, D_1 denote the corresponding disk bundles with boundary E then $S^{n+1} = D_0 \cup_E D_1$.

Fix $p \geq 0$ and consider the Serre spectral sequence for the fibration $S^k \rightarrow E \rightarrow B_0$ with coefficients in \mathbf{F}_p . If this fails to collapse then $H^k(\pi_0): H^k(B_0; \mathbf{F}_p) \rightarrow H^k(E; \mathbf{F}_p)$ is surjective. Since $l > k$ it is always true that $H^k(\pi_1)$ is surjective. Choose classes $\alpha \in H^k(B_0; \mathbf{F}_p)$, $\beta \in H^k(B_1; \mathbf{F}_p)$ mapping to the same non-zero class in $H^k(E; \mathbf{F}_p)$. The Mayer-Vietoris sequence for the decomposition $S^{n+1} = D_0 \cup_E D_1$ then gives a class $\gamma \in H^k(S^{n+1}; \mathbf{F}_p)$ restricting to α and β , which is absurd.

Thus the spectral sequence for $S^k \rightarrow E \rightarrow B_0$ collapses and so $H_*(B_0; \mathbf{F}_p) \cong H_*(S^l \times S^{l+k}; \mathbf{F}_p)$. Using Poincaré duality for B_0 we see that $H^*(B_0; \mathbf{F}_p)$ and $H^*(S^l \times S^{l+k}; \mathbf{F}_p)$ are isomorphic as graded algebras. Thus B_0 is elliptic by 3.4 and E is elliptic by 3.3.

3.6. *Simply connected closed manifolds M with a smooth action by a compact Lie group G , having a simply connected codimension one orbit.*

Here we may assume G is connected. Let the orbit be G/K , and convert the inclusion of G/K into a fibration $F \rightarrow G/K \rightarrow M$. From [9; Table 1.5] we see that for any p , $\dim H_i(F; \mathbf{F}_p) \leq 2$, all i . Thus applying the Serre spectral sequence to the fibration $\Omega(G/K) \rightarrow \Omega M \rightarrow F$ and using 3.1 for G/K we see that $H_*(\Omega M; \mathbf{F}_p)$ grows polynomially.

3.7. *Simply connected manifolds $M \# N$ with each of the rings $H^*(M; \mathbf{Z})$, $H^*(N; \mathbf{Z})$ generated by a single class.*

By Van Kampen's theorem both M and N are simply connected, and so their fundamental cohomology classes are not torsion. Since each ring is monogenic, $H^*(M; \mathbf{Z})$ and $H^*(N; \mathbf{Z})$ are torsion free. Thus $H^*(M; \mathbf{F}_p)$ and $H^*(N; \mathbf{F}_p)$ are also monogenic, and so $H^*(M \# N; \mathbf{F}_p)$ is doubly generated. Now apply 3.4.

REFERENCES

- [1] BROWDER, W. On differential Hopf algebras. *Trans. Amer. Math. Soc.* 107 (1963), 153-176.
- [2] FELIX, Y., S. HALPERIN and J.-C. THOMAS. The homotopy Lie algebra for finite complexes. *Publ. de l'Institut des Hautes Etudes Scientifiques* 56 (1983), 387-410.
- [3] FELIX, Y., S. HALPERIN, J.-M. LEMAIRE and J.-C. THOMAS. Mod p loop space homology. *Invent. Math.* 95 (1989), 247-262.
- [4] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Hopf algebras of polynomial growth. *J. of Algebra* 125 (1989), 408-417.
- [5] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Loop space homology of spaces of LS category one and two. *Math. Ann.* 287 (1990), 377-387.

- [6] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Elliptic spaces. *Bulletin Amer. Math. Soc.* 25 (1991), 69-73.
- [7] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Elliptic Hopf algebras. *J. London Math. Soc.* 43 (1991), 535-545.
- [8] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Loop space homology at large primes. In preparation.
- [9] GROVE, K. and S. HALPERIN. Dupin hypersurfaces, group actions and the double mapping cylindre. *J. diff. Geom.* 26 (1987), 429-459.
- [10] HARDY, G.H. and S. RAMANUJAN. Asymptotic formulae in Combinatory Analysis. *Proc. London Math. Soc.* 17 (1918), 75-115.
- [11] TATE, J. Homology of Noetherian rings and local rings. *Ill. J. Math.* 1 (1957), 14-25.
- [12] WIEBE, H. Über homologische Invarianten lokaler Ringe. *Math. Ann.* 179 (1969), 257-274.

(Reçu le 14 février 1992)

Yves Felix

Institut de Mathématiques
Université Catholique de Louvain
B-1348 Louvain-La-Neuve, Belgique

Stephen Halperin

Department of Mathematics
Scarborough College, University of Toronto
Scarborough, Canada M1C 1A4

Jean-Claude Thomas

U.F.R. de Mathématiques Pures et Appliquées
Université des Sciences et Technologie de Lille
59655 Villeneuve d'Ascq, France