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(to compare this inequality to the inequality as originally stated by Zimmert one has to apply the gamma duplication formula). He chose $s = 2$ to obtain

$$\frac{R}{w} \geq 0.02 \cdot \exp(0.46r_1 + 0.1r_2).$$

Zimmert deduced his regulator bounds by an ingenious, but quite involved, investigation of certain analytic properties of the partial Dedekind zeta function associated to the class of principal ideals of K .

In this note we show that it is possible to deduce the above theorem by a simple estimate from a certain, almost obvious, monotonicity property of Hecke's theta function associated to the maximal order of K (see below). Moreover, we indicate below how this method of proof can be refined to yield exactly Zimmert's bounds. The technique of estimating which we apply is a sort of simple variation of a method which is developed in [F-S] to obtain lower bounds for L^p -norms of a certain class of functions. It was found during a careful analysis of Zimmert's method and reflects, though it looks much easier, still very much the spirit of Zimmert's original proof.

2. PROOF. Let $|\cdot|_j$ for $1 \leq j \leq r := r_1 + r_2$ denote the archimedean absolute values of K , let G denote the r -fold direct product of the multiplicative group of the positive reals \mathbf{R}_+ , and let V denote the image in G of the units of K under the map

$$\eta \mapsto (\dots, |\eta|_j^{n_j}, \dots),$$

where n_j equals 1 or 2 accordingly as $|\cdot|_j$ is real or complex. Denote by δ the group homomorphism

$$\delta: G \rightarrow \mathbf{R}_+, \quad \delta((\dots, x_j, \dots)) = x_1 \cdots x_r.$$

Its kernel contains V , and by Dirichlet's unit theorem $\ker \delta/V$ is compact. We can thus fix a Haar measure μ on G/V by requiring

$$\int_{G/V} g \circ \delta d\mu = \frac{R}{w} \int_0^\infty g(t) \frac{dt}{t}$$

for any integrable function on \mathbf{R}_+ . Let

$$Z(s) := \gamma(s) \sum_{\alpha \in \mathfrak{R}} |N_{K/\mathbf{Q}} \alpha|^{-s},$$

where \mathfrak{R} is a set of representatives for the non-zero elements of \mathfrak{O} , the ring

of integers in K , modulo units. According to Hecke [H] (and according to the choice of μ) one has, for $\operatorname{Re}(s) > 1$, the integral representation

$$Z(s) = \int_{G/V} \Theta \delta^s d\mu .$$

Here Θ is a smooth, non-negative and V -invariant function on G , which is given by

$$\Theta(x) = \sum_{\substack{\alpha \in \mathfrak{D} \\ \alpha \neq 0}} \exp \left(- \sum_{j=1}^r |\alpha|_j^2 x_j^{2/n_j} \right) .$$

The main observation for the proof of the theorem is the

LEMMA. *The function $(1 + \Theta)\delta$ is increasing in each argument.*

Proof. This follows from Hecke's theta formula [H, p. 165-166]

$$(1 + \Theta(x)) \delta(x) = \frac{\pi^{\frac{n}{2}} 2^{r_2}}{\sqrt{|d|}} \sum_{\alpha \in \mathfrak{D}^{-1}} \exp \left(- \pi^2 \sum_{j=1}^r n_j^2 |\alpha|_j^2 x_j^{-2/n_j} \right) ,$$

i.e. by applying Poisson summation to the series defining $\Theta(x)$ (here \mathfrak{D} and d denote the different and discriminant of K). \square

We can now give the

Proof of the theorem. For $a \in \mathbf{R}_+$ set

$$I(a) := \int_{G/V} (1 + \Theta(x)) \delta(x) w((a\delta(x))^{s-1}) d\mu(x) ,$$

where we use

$$w(t) = t \max(0, \log(1/t)) ,$$

and where $s > 1$ as in the theorem.

For any $\varepsilon > 0$, one has $w(t) = O(t^\varepsilon)$ and $|w(t+h) - w(t)| / |h| \leq |w'(t)| = O(t^{-\varepsilon})$ as $t \rightarrow 0$. Thus, using the convergence of the integral representation of $Z(s)$ for $s > 1$, we deduce that the integral defining $I(a)$ is finite, and, on applying Lebesgue's theorem, that $I(a)$ is differentiable and its derivative is obtained by differentiating under the integral sign. Here we agree to use $w'(1)$ for the derivative on the right, i.e. $w'(1) = 0$.

On replacing x by $x/a^{1/r}$ in the integral defining $I(a)$ we deduce from the lemma that $I(a)$ is decreasing. Hence $I'(a) \leq 0$, from which we obtain, writing $\sigma = s - 1$,

$$\begin{aligned} & -\frac{d}{da} \int_{G/V} \delta w((a\delta)^\sigma) d\mu \geq \frac{d}{da} \int_{G/V} \Theta \delta w((a\delta)^\sigma) d\mu \\ & = \sigma a^{\sigma-1} \int_{G/V} \Theta \delta^s w'((a\delta)^\sigma) d\mu \geq -\sigma a^{\sigma-1} \int_{G/V} \Theta \delta^s (1 + \log(a\delta)^\sigma) d\mu \\ & = -\sigma^2 a^{\sigma-1} Z(s) \left(\frac{1}{\sigma} + \log a + \frac{Z'(s)}{Z} \right). \end{aligned}$$

By the choice of μ the left-hand side equals $R\sigma/(ws^2a^2)$. Multiplying the above inequality by s^2a^2/σ and then maximizing the right hand side, i.e. choosing

$$\frac{1}{\sigma} + \log a + \frac{Z'(s)}{Z} = -\frac{1}{s},$$

we find

$$(1) \quad \frac{R}{w} \geq \frac{s(s-1)}{e} \exp\left(-\frac{s}{s-1}\right) Z(s) \exp\left(-s \frac{Z'(s)}{Z}\right).$$

Finally, $Z(s) \geq \gamma(s)$, since the Dirichlet series $D(s)$ in the definition of $Z(s)$ satisfies $D(s) > 1$, and $\frac{Z'(s)}{Z} \leq \frac{\gamma'(s)}{\gamma}$, since $D'(s) < 0$. Thus, (1) implies the claimed inequality. \square

3. CONCLUDING REMARKS. To obtain a lower bound as sharp as Zimmert's one can proceed as above, but with a variant Θ_1 of the function Θ . Namely, fix a real number $s > 1$, and define $\Theta_1(x)$ by the same series as $\Theta(x)$ but with the term

$$\exp\left(-\sum |\alpha|_j x_j^{2/n_j}\right)$$

replaced by

$$\prod_{j=1}^r f_j(|\alpha|_j^{n_j} x_j),$$