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The following questions therefore are suggestive:

- 1) if one starts with an arbitrary  $F \in M_{1/2, k-1/2}(\Gamma_2)$ , does the above limit process produce skew-holomorphic Jacobi forms of weight  $k$ ?
- 2) define  $M_{1/2, k-1/2}^*(\Gamma_2)$  as the subspace of  $M_{1/2, k-1/2}(\Gamma_2)$  consisting of the intersection of the kernels of the operators  $\mathcal{E}_p$  for all primes  $p$ . Does there exist a natural map  $V$  from skew-holomorphic Jacobi forms of weight  $k$  and index 1 to  $M_{1/2, k-1/2}^*(\Gamma_2)$  similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on  $\mathrm{Sp}_2$ . It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus  $n > 2$  has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a “Maass space” eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for  $n \geq 33$  the map which sends a Siegel modular form of weight 16 on  $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$  to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for  $n = 3$  due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform  $F$  of even integral weight  $k$  on  $\Gamma_3$  could be constructed from a pair  $(f, g)$  of elliptic Hecke eigenforms of weights  $(k_1, k_2)$  equal to  $(k, 2k - 4)$  or  $(k - 2, 2k - 2)$  such that the (formal) spinor zeta function of  $F$  should be equal to  $L_f(s - k_2/2) L_f(s - k_2/2 + 1) L_{f \otimes g}(s)$  where  $L_{f \otimes g}(s)$  essentially is the Rankin convolution of  $f$  and  $g$  ([*loc. cit.*, §4]; note that for  $n > 2$  the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on  $\Gamma_n$  is not known).

### §3. SPINOR ZETA FUNCTIONS

#### 3.1. RESULTS

Although the Maass space  $S_k^*(\Gamma_2)$  as discussed in the previous section is an important subspace of  $S_k(\Gamma_2)$  in its own right, one quickly realizes that the “true” Siegel cusp forms on  $\Gamma_2$  should lie in the orthogonal complement of  $S_k^*(\Gamma_2)$  (cf. Theorem 2 in §2 and its discussion). Is it therefore even more

surprising that forms in the Maass space can be used to study forms in  $S_k^*(\Gamma_2)^\perp$  (in fact, spinor zeta functions of Hecke eigenforms in  $S_k^*(\Gamma_2)^\perp$ ). Thus the importance of the Maass space seems to go much beyond that what is expected from §2.

Let  $F$  and  $G$  be Siegel cusp forms of integral weight  $k$  on  $\Gamma_2$ . Denote by  $\phi_m$  and  $\psi_m$  ( $m \geq 1$ ) the Fourier-Jacobi coefficients of  $F$  and  $G$ , respectively and define a formal Dirichlet series of Rankin-type by

$$(6) \quad D_{F,G}(s) := \zeta(2s - 2k + 4) \sum_{m \geq 1} \langle \phi_m, \psi_m \rangle m^{-s}$$

(this series was introduced by Skoruppa and the author in [18]).

A variant of the classical Hecke argument shows that  $\langle \phi_m, \psi_m \rangle \ll_{F,G} m^k$  so that  $D_{F,G}(s)$  is absolutely convergent for  $\operatorname{Re}(s) > k + 1$ . We put

$$D_{F,G}^*(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,G}(s) \quad (\operatorname{Re}(s) > k + 1).$$

THEOREM 1 [18]. *The function  $D_{F,G}(s)$  has a meromorphic continuation to  $\mathbb{C}$  which is holomorphic except for a possible simple pole of residue*

$$\frac{4^k \pi^{k+2}}{(k-2)!} \langle F, G \rangle$$

at  $s = k$ . Furthermore, the functional equation

$$D_{F,G}^*(2k - 2 - s) = D_{F,G}^*(s)$$

holds.

THEOREM 2 [18]. *Let  $k$  be even. Let  $F \in S_k(\Gamma_2)$  be a Hecke eigenform and  $G$  be a function in the Maass space  $S_k^*(\Gamma_2)$ . Then*

$$D_{F,G}(s) = \langle \phi_1, \psi_1 \rangle Z_F(s).$$

The proof of Theorem 1 is based on the Rankin-Selberg method applied with an Eisenstein series of Klingen-type on  $\operatorname{Sp}_2$ . The proof of Theorem 2 uses Theorem 1 of §2 applied with  $\phi$  a Poincaré series; furthermore, an explicit formula for the action on Fourier coefficients of the operator  $V_m^*$  adjoint to  $V_m$  w.r.t the Petersson scalar products and the relations due to Andrianov [1, Chap. 2] between eigenvalues and Fourier coefficients of Hecke eigenforms play an important role. Let us mention that Theorem 2 could also be deduced from results of Gritsenko [13, p. 266].

In [38], Yamazaki using the theory of Eisenstein series à la Langlands studied the analytic properties of generalizations to arbitrary genus  $n$  of the

series (6). Recently, Krieg [24] gave a more elementary proof of (some of) the results of [38] using well-known properties of Epstein zeta functions. However, it is clear from the  $\Gamma$ -factors and the type of the functional equations that for  $n > 2$  there cannot be any direct connection between the series studied in [24, 38] and spinor zeta functions.

## 1.2 PROBLEMS

i) Suppose that  $k$  is even. If  $F$  is a non-zero Hecke eigenform in  $S_k(\Gamma_2)$ , is  $\phi_1 \neq 0$ ? (This question was already asked in [33].) The answer is positive for  $k \leq 32$  as numerical computations due to Skoruppa [35] show. Note that by Theorem 2 a positive answer gives a new proof for the analytic continuation and the functional equation of  $Z_F(s)$ .

ii) Let  $F$  be a Hecke eigenform in  $S_k(\Gamma_2)$ . The only critical point of  $Z_F(s)$  in Deligne's sense is  $s = k - 1$ , i.e. the center of symmetry of the functional equation as is easily checked. Conjecturally therefore  $Z_F(k - 1)$  should be equal to the determinant of a "period matrix" times an algebraic number (one may suppose that  $k$  is even since otherwise  $Z_F(k - 1) = 0$  as follows from the sign in the functional equation). To the author's knowledge, nothing so far in this direction has been proved. Could Theorem 2 eventually be useful in this context?

As a side remark, let us mention here that Böcherer [4] motivated by Waldspurger's results [37] about the central critical values of quadratic twists of Hecke  $L$ -functions of elliptic Hecke eigenforms, for  $k$  even has conjectured that the central critical value of the twist of  $Z_F(s)$  by a quadratic Dirichlet character of conductor  $D < 0$  should be proportional to the *square* of

$\sum_{\{T > 0\} / \sim, \text{disc } T = D} a(T)$  where  $a(T)$  are the Fourier coefficients of  $F$  and the

sum is over a set of  $\Gamma_1$ -representatives of positive definite integral binary quadratic forms  $T$  of discriminant  $D$ . This conjecture is true if  $F$  is in the Maass space as follows from Theorem 2 in §2 in connection with Waldspurger's results, cf. [4]. The conjecture when generalized to level  $> 1$  is also true if the corresponding form has weight 2 and is the Yoshida lift of two elliptic cusp forms [6].

iii) Let  $F$  be a cuspidal Hecke eigenform and assume that  $F$  is in  $S_k^*(\Gamma_2)^\perp$  if  $k$  is even. Does the function  $D_{F,F}(s)$  have any intrinsic arithmetical meaning? (This question was already asked in [33], too; note that  $D_{F,F}(s)$  for  $F$  as above cannot be proportional to  $Z_F(s)$  since  $D_{F,F}(s)$  has a pole at  $s = k$  while  $Z_F(s)$  is holomorphic there, cf. §2). For some numerical computations in this direction in the case  $k = 20$  (the first case where  $S_k^*(\Gamma_2)^\perp \neq \{0\}$ ) we refer to [23].