

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS
Autor: Kohnen, Winfried
Kapitel: §3. Spinor zeta functions
DOI: <https://doi.org/10.5169/seals-60416>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The following questions therefore are suggestive:

- 1) if one starts with an arbitrary $F \in M_{1/2, k-1/2}(\Gamma_2)$, does the above limit process produce skew-holomorphic Jacobi forms of weight k ?
- 2) define $M_{1/2, k-1/2}^*(\Gamma_2)$ as the subspace of $M_{1/2, k-1/2}(\Gamma_2)$ consisting of the intersection of the kernels of the operators \mathcal{E}_p for all primes p . Does there exist a natural map V from skew-holomorphic Jacobi forms of weight k and index 1 to $M_{1/2, k-1/2}^*(\Gamma_2)$ similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on Sp_2 . It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus $n > 2$ has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for $n \geq 33$ the map which sends a Siegel modular form of weight 16 on $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$ to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for $n = 3$ due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform F of even integral weight k on Γ_3 could be constructed from a pair (f, g) of elliptic Hecke eigenforms of weights (k_1, k_2) equal to $(k, 2k - 4)$ or $(k - 2, 2k - 2)$ such that the (formal) spinor zeta function of F should be equal to $L_f(s - k_2/2)L_f(s - k_2/2 + 1)L_{f \otimes g}(s)$ where $L_{f \otimes g}(s)$ essentially is the Rankin convolution of f and g ([*loc. cit.*, §4]; note that for $n > 2$ the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on Γ_n is not known).

§3. SPINOR ZETA FUNCTIONS

3.1. RESULTS

Although the Maass space $S_k^*(\Gamma_2)$ as discussed in the previous section is an important subspace of $S_k(\Gamma_2)$ in its own right, one quickly realizes that the "true" Siegel cusp forms on Γ_2 should lie in the orthogonal complement of $S_k^*(\Gamma_2)$ (cf. Theorem 2 in §2 and its discussion). It is therefore even more

surprising that forms in the Maass space can be used to study forms in $S_k^*(\Gamma_2)^\perp$ (in fact, spinor zeta functions of Hecke eigenforms in $S_k^*(\Gamma_2)^\perp$). Thus the importance of the Maass space seems to go much beyond that what is expected from §2.

Let F and G be Siegel cusp forms of integral weight k on Γ_2 . Denote by ϕ_m and ψ_m ($m \geq 1$) the Fourier-Jacobi coefficients of F and G , respectively and define a formal Dirichlet series of Rankin-type by

$$(6) \quad D_{F,G}(s) := \zeta(2s - 2k + 4) \sum_{m \geq 1} \langle \phi_m, \psi_m \rangle m^{-s}$$

(this series was introduced by Skoruppa and the author in [18]).

A variant of the classical Hecke argument shows that $\langle \phi_m, \psi_m \rangle \ll_{F,G} m^k$ so that $D_{F,G}(s)$ is absolutely convergent for $\operatorname{Re}(s) > k + 1$. We put

$$D_{F,G}^*(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,G}(s) \quad (\operatorname{Re}(s) > k + 1).$$

THEOREM 1 [18]. *The function $D_{F,G}(s)$ has a meromorphic continuation to \mathbf{C} which is holomorphic except for a possible simple pole of residue*

$$\frac{4^k \pi^{k+2}}{(k-2)!} \langle F, G \rangle$$

at $s = k$. Furthermore, the functional equation

$$D_{F,G}^*(2k - 2 - s) = D_{F,G}^*(s)$$

holds.

THEOREM 2 [18]. *Let k be even. Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform and G be a function in the Maass space $S_k^*(\Gamma_2)$. Then*

$$D_{F,G}(s) = \langle \phi_1, \psi_1 \rangle Z_F(s).$$

The proof of Theorem 1 is based on the Rankin-Selberg method applied with an Eisenstein series of Klingen-type on Sp_2 . The proof of Theorem 2 uses Theorem 1 of §2 applied with ϕ a Poincaré series; furthermore, an explicit formula for the action on Fourier coefficients of the operator V_m^* adjoint to V_m w.r.t the Petersson scalar products and the relations due to Andrianov [1, Chap. 2] between eigenvalues and Fourier coefficients of Hecke eigenforms play an important role. Let us mention that Theorem 2 could also be deduced from results of Gritsenko [13, p. 266].

In [38], Yamazaki using the theory of Eisenstein series à la Langlands studied the analytic properties of generalizations to arbitrary genus n of the

series (6). Recently, Krieg [24] gave a more elementary proof of (some of) the results of [38] using well-known properties of Epstein zeta functions. However, it is clear from the Γ -factors and the type of the functional equations that for $n > 2$ there cannot be any direct connection between the series studied in [24, 38] and spinor zeta functions.

1.2 PROBLEMS

i) Suppose that k is even. If F is a non-zero Hecke eigenform in $S_k(\Gamma_2)$, is $\phi_1 \neq 0$? (This question was already asked in [33].) The answer is positive for $k \leq 32$ as numerical computations due to Skoruppa [35] show. Note that by Theorem 2 a positive answer gives a new proof for the analytic continuation and the functional equation of $Z_F(s)$.

ii) Let F be a Hecke eigenform in $S_k(\Gamma_2)$. The only critical point of $Z_F(s)$ in Deligne's sense is $s = k - 1$, i.e. the center of symmetry of the functional equation as is easily checked. Conjecturally therefore $Z_F(k - 1)$ should be equal to the determinant of a "period matrix" times an algebraic number (one may suppose that k is even since otherwise $Z_F(k - 1) = 0$ as follows from the sign in the functional equation). To the author's knowledge, nothing so far in this direction has been proved. Could Theorem 2 eventually be useful in this context?

As a side remark, let us mention here that Böcherer [4] motivated by Waldspurger's results [37] about the central critical values of quadratic twists of Hecke L -functions of elliptic Hecke eigenforms, for k even has conjectured that the central critical value of the twist of $Z_F(s)$ by a quadratic Dirichlet character of conductor $D < 0$ should be proportional to the *square* of

$\sum_{\{T > 0\} / \sim, \text{disc } T = D} a(T)$ where $a(T)$ are the Fourier coefficients of F and the

sum is over a set of Γ_1 -representatives of positive definite integral binary quadratic forms T of discriminant D . This conjecture is true if F is in the Maass space as follows from Theorem 2 in §2 in connection with Waldspurger's results, cf. [4]. The conjecture when generalized to level > 1 is also true if the corresponding form has weight 2 and is the Yoshida lift of two elliptic cusp forms [6].

iii) Let F be a cuspidal Hecke eigenform and assume that F is in $S_k^*(\Gamma_2)^\perp$ if k is even. Does the function $D_{F,F}(s)$ have any intrinsic arithmetical meaning? (This question was already asked in [33], too; note that $D_{F,F}(s)$ for F as above cannot be proportional to $Z_F(s)$ since $D_{F,F}(s)$ has a pole at $s = k$ while $Z_F(s)$ is holomorphic there, cf. §2). For some numerical computations in this direction in the case $k = 20$ (the first case where $S_k^*(\Gamma_2)^\perp \neq \{0\}$) we refer to [23].