

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 39 (1993)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS  
**Autor:** Kohnen, Winfried  
**Kapitel:** 2.2. Problems  
**DOI:** <https://doi.org/10.5169/seals-60416>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 14.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

cients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

## 2.2. PROBLEMS

i) Since for fixed  $k$  the dimension of  $J_{k,m}$  grows linearly in  $m$ , the map  $\rho_m$  defined by (3) for  $m \gg_k 0$  cannot be surjective. Is there any simple or nice description of the image of  $\rho_m$  or  $(\text{im } \rho_m | S_k(\Gamma_2))^\perp$ ? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on  $\Gamma_2$  which generate  $S_k(\Gamma_2)$ , as certain infinite linear combinations of Poincaré series on  $\Gamma_1^J$  [22]. Taking scalar products one obtains a characterization of  $(\text{im } \rho_m | S_k(\Gamma_2))^\perp$  as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that  $\rho_1$  is surjective).

ii) A skew-holomorphic Jacobi form of weight  $k \in \mathbb{Z}$  and index  $m \in \mathbb{N}_0$  on  $\Gamma_1^J$  as introduced by Skoruppa is a complex-valued  $C^\infty$ -function  $\phi(\tau, z)$  ( $\tau \in \mathcal{H}$ ,  $z \in \mathbb{C}$ ) satisfying the following properties: 1)  $\phi$  is holomorphic in  $z$  and is annihilated by the heat operator  $8\pi i m \partial/\partial \tau - \partial^2/\partial z^2$ ; 2)  $\phi$  satisfies the same transformation formula under  $\Gamma_1^J$  as a holomorphic Jacobi form of weight  $k$  and index  $m$  (cf. § 1.2) except that the factor  $(c\tau + d)^k$  has to be replaced by  $(c\bar{\tau} + d)^{k-1} |c\tau + d|$ ; 3)  $\phi$  has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbb{Z}, r^2 \geq 4mn} c(n, r) \exp \left( -\pi \frac{r^2 - 4mn}{m} v \right) q^n \zeta^r \quad (v = \text{Im}(\tau)).$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of  $F(Z)$  w.r.t.  $e^{2\pi i \tau'}$  (where as usual  $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ ) not only depend on  $\tau$  and  $z$  but also on  $\text{Im}(\tau')$ , and it is a priori not obvious how to get rid of the latter variable and to produce “true” Jacobi forms.

Let  $k$  be an odd integer and denote by  $M_{1/2, k-1/2}(\Gamma_2)$  the space of Siegel-Maass wave forms “of type  $(1/2, k-1/2)$ ” as defined in [26], i.e. the space of real-analytic functions  $F: \mathcal{H}_2 \rightarrow \mathbb{C}$  which satisfy

$$F(M \langle Z \rangle) = \det(C\bar{Z} + D)^{k-1} | \det(CZ + D) | F(Z)$$

for all  $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$  and which are annihilated by the matrix differential operator

$$\Omega_{1/2, k-1/2} := (Z - \bar{Z}) \left( (Z - \bar{Z}) \frac{\partial}{\partial Z} \right)' \frac{\partial}{\partial \bar{Z}} + \frac{1}{2} (Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}} - \left( k - \frac{1}{2} \right) (Z - \bar{Z}) \frac{\partial}{\partial Z}$$

$$\text{where } \frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial \tau} & \frac{1}{2} \frac{\partial}{\partial z} \\ \frac{1}{2} \frac{\partial}{\partial \bar{z}} & \frac{\partial}{\partial \tau'} \end{pmatrix}$$

and  $\frac{\partial}{\partial \bar{Z}}$  is defined analogously (the notation “of type  $(1/2, k-1/2)$ ” comes from the fact that the factor of automorphy of  $F$  can be written as  $\det(C\bar{Z} + D)^{k-1/2} \det(CZ + D)^{1/2}$  with appropriate choice of the square root).

Using certain invariance properties of  $\Omega_{1/2, k-1/2}$  under the action of  $\text{Sp}_2(\mathbf{R})$  one can define Hecke operators  $T_n (n \in \mathbf{N})$  on  $M_{1/2, k-1/2}(\Gamma_2)$  in the usual way. Let

$$E_{1/2, k-1/2}^{(2)}(Z) := \sum_{(C, D)} \det(CZ + D)^{-k+1} |\det((CZ + D))|^{-1} \quad (k > 3)$$

be the Maass-Siegel-Eisenstein series in  $M_{1/2, k-1/2}(\Gamma_2)$  ([26; 27, §18]; summation over all pairs  $(C, D)$  of relatively prime symmetric  $(2, 2)$ -matrices inequivalent under left-multiplication by  $GL_2(\mathbf{Z})$ ). Then the following can be shown:

- 1) The function  $E_{1/2, k-1/2}^{(2)}$  is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to  $\zeta(s-k+1) \zeta(s-k+2) L_{E_{2k-2}}(s)$  where  $E_{2k-2}$  is the normalized Eisenstein series of weight  $2k-2$  on  $\Gamma_1$  (this implies that  $E_{1/2, k-1/2}^{(2)}$  for all primes  $p$  is annihilated by the Hecke operator  $\mathcal{E}_p$  defined analogously as in (5));
- 2) if  $e_{1/2, k-1/2; m}(\tau, z, \text{Im}(\tau'))$  is the  $m$ -th Fourier-Jacobi coefficient of  $E_{1/2, k-1/2}^{(2)}$  and if for  $m > 0$  one carries out a similar limit process as in [19, §2, Remark ii) after the proof of Thm. 1], i.e. essentially replaces  $\text{Im}(\tau')$  by  $(\text{Im}(z))^2 / \text{Im}(\tau) + \delta$  and lets  $\delta \rightarrow \infty$ , then one obtains a skew-holomorphic Eisenstein series of weight  $k$  and index  $m$  (in fact, finite linear combinations of such Eisenstein series if  $m$  is not squarefree).

The following questions therefore are suggestive:

- 1) if one starts with an arbitrary  $F \in M_{1/2, k-1/2}(\Gamma_2)$ , does the above limit process produce skew-holomorphic Jacobi forms of weight  $k$ ?
- 2) define  $M_{1/2, k-1/2}^*(\Gamma_2)$  as the subspace of  $M_{1/2, k-1/2}(\Gamma_2)$  consisting of the intersection of the kernels of the operators  $\mathcal{E}_p$  for all primes  $p$ . Does there exist a natural map  $V$  from skew-holomorphic Jacobi forms of weight  $k$  and index 1 to  $M_{1/2, k-1/2}^*(\Gamma_2)$  similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on  $\mathrm{Sp}_2$ . It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus  $n > 2$  has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a “Maass space” eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for  $n \geq 33$  the map which sends a Siegel modular form of weight 16 on  $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$  to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for  $n = 3$  due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform  $F$  of even integral weight  $k$  on  $\Gamma_3$  could be constructed from a pair  $(f, g)$  of elliptic Hecke eigenforms of weights  $(k_1, k_2)$  equal to  $(k, 2k - 4)$  or  $(k - 2, 2k - 2)$  such that the (formal) spinor zeta function of  $F$  should be equal to  $L_f(s - k_2/2) L_f(s - k_2/2 + 1) L_{f \otimes g}(s)$  where  $L_{f \otimes g}(s)$  essentially is the Rankin convolution of  $f$  and  $g$  ([*loc. cit.*, §4]; note that for  $n > 2$  the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on  $\Gamma_n$  is not known).

### §3. SPINOR ZETA FUNCTIONS

#### 3.1. RESULTS

Although the Maass space  $S_k^*(\Gamma_2)$  as discussed in the previous section is an important subspace of  $S_k(\Gamma_2)$  in its own right, one quickly realizes that the “true” Siegel cusp forms on  $\Gamma_2$  should lie in the orthogonal complement of  $S_k^*(\Gamma_2)$  (cf. Theorem 2 in §2 and its discussion). It is therefore even more