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only on the discriminant  $D := r^2 - 4mn$  and the residue class  $r \pmod{2m}$ .

The Petersson scalar product on  $J_{k,m}^{\text{cusp}}$  is normalized by

$$\langle \phi, \psi \rangle = \int_{\Gamma_1^J \backslash \mathcal{H} \times \mathbb{C}} \phi(\tau, z) \overline{\psi(\tau, z)} \exp(-4\pi m y^2 / \nu) \nu^{k-3} du dv dx dy$$

$$(\tau = u + iv, z = x + iy).$$

For basic facts about Jacobi forms we refer to [9].

## §2. THE MAASS SPACE

### 2.1. RESULTS

Let  $F$  be a Siegel modular form of integral weight  $k$  on  $\Gamma_2$  and write the Fourier expansion of  $F$  in the form

$$(1) \quad F(Z) = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \quad \left( Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2 \right).$$

Using the injection

$$(2) \quad \Gamma_1^J \rightarrow \Gamma_2, \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa) \right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and the transformation formula of  $F$  it is easy to see that the functions  $\phi_m$  are in  $J_{k,m}$ . The expansion (1) is referred to as the Fourier-Jacobi expansion of  $F$ .

Thus for any  $m \in \mathbf{N}_0$  we obtain a linear map

$$(3) \quad \rho_m : M_k(\Gamma_2) \rightarrow J_{k,m}, \quad F \mapsto \phi_m.$$

Note that  $\rho_0$  is equal to the Siegel  $\Phi$ -operator.

We shall be interested in the case  $m = 1$ . For  $k$  odd,  $\rho_1$  is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For  $k$  even,  $\rho_1$  was studied in detail by Maass [28, 29] who showed the existence of a natural map  $V : J_{k,1} \rightarrow M_k(\Gamma_2)$  such that the composite  $\rho_1 \circ V$  is the identity. More precisely, let  $\phi \in J_{k,1}$  with Fourier coefficients  $c(n, r)$  ( $n, r \in \mathbf{Z}; r^2 \leq 4n$ ) and for  $m \in \mathbf{N}_0$  define

$$(4) \quad (V_m \phi)(\tau, z) := \sum_{n, r \in \mathbf{Z}, r^2 \leq 4mn} \left( \sum_{d | (n, r, m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right) \right) q^n \zeta^r$$

(if  $m = 0$ , the term  $\sum_{d|0} d^{k-1} c(0, 0)$  on the right of (4) has to be interpreted

as  $\frac{1}{2} \zeta(1-k)$ ; note that  $V_1 \phi = \phi$ ). Using a more invariant definition

of  $V_m$  in terms of the action of a set of representatives for  $\Gamma_1 \setminus \{M \in \mathbf{Z}^{(2,2)} \mid \det M = m\}$  one checks that  $V_m \phi \in J_{k,m}$  [9, §4]. Put

$$(V\phi)(Z) := \sum_{m \geq 0} (V_m \phi)(\tau, z) e^{2\pi i m \tau'} \quad \left( Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2 \right).$$

We denote by  $T_n (n \in \mathbf{N})$  the usual Hecke operators on  $M_k(\Gamma_2)$  resp.  $S_k(\Gamma_2)$  [12, IV; 1, II]; thus, if  $p$  is a prime,  $T_p$  resp.  $T_{p^2}$  correspond to the two generators

$$\Gamma_2 \begin{pmatrix} 1_2 & 0 \\ 0 & p1_2 \end{pmatrix} \Gamma_2 \text{ resp. } \Gamma_2 \text{diag}(1, p, p^2, p) \Gamma_2$$

of the local Hecke algebra of  $\Gamma_2$  at  $p$ . We denote by  $T_{J,n} (n \in \mathbf{N})$  the Hecke operators on  $J_{k,m}$  resp.  $J_{k,m}^{\text{cusp}}$  [9, §4].

**THEOREM 1.** (Maass [28, 29], Andrianov [2]). *Suppose that  $k$  is even. The map  $\phi \mapsto V\phi$  gives an injection  $J_{k,1} \rightarrow M_k(\Gamma_2)$  which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. If  $p$  is a prime, one has  $T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$  and  $T_{p^2} \circ V = V \circ (T_{J,p^2}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2})$ .*

The image of  $J_{k,1}$  under  $V$  is called the Maass space and will be denoted by  $M_k^*(\Gamma_2)$ . One knows that  $M_k^*(\Gamma_2) = \mathbf{C}E_k^{(2)} \oplus S_k^*(\Gamma_2)$  where  $E_k^{(2)}$  is the Siegel-Eisenstein series of weight  $k$  on  $\Gamma_2$  and  $S_k^*(\Gamma_2) := M_k^*(\Gamma_2) \cap S_k(\Gamma_2)$ . Observe that  $\dim M_k^*(\Gamma_2) = \dim J_{k,1}$  grows linearly in  $k$  while  $\dim M_k(\Gamma_2)$  grows like  $k^3$ .

Note that Theorem 1 implies that  $M_k^*(\Gamma_2)$  is stable under all Hecke operators and that it is annihilated by the operator

$$(5) \quad \mathcal{C}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2},$$

for every prime  $p$ .

Let  $F \in M_k(\Gamma_2)$  be a non-zero Hecke eigenform and denote by  $\lambda_n (n \in \mathbf{N})$  its eigenvalues under  $T_n$ . If  $p$  is a prime, we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3}X^3 + p^{4k-6}X^4$$

so that  $Z_{F,p}(p^{-s})$  ( $s \in \mathbf{C}$ ) is the local spinor zeta function of  $F$  at  $p$ . We put

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s}) \quad (\operatorname{Re}(s) \geq 0).$$

One has

$$Z_F(s) = \zeta(2s - 2k + 4)^{-1} \sum_{n \geq 1} \lambda_n n^{-s} \quad (\operatorname{Re}(s) \geq 0).$$

If  $F$  is an Eisenstein series, then it is well-known that  $Z_F(s)$  can be expressed in terms of products of Hecke  $L$ -functions of elliptic modular forms.

Suppose that  $F$  is cuspidal. Then it was proved in [1, Chap. 3] that  $Z_F(s)$  has a meromorphic continuation to  $\mathbf{C}$  which is holomorphic everywhere if  $k$  is odd and is holomorphic except for a possible simple pole at  $s = k$  if  $k$  is even. Moreover, the global function  $Z_F^*(s) := (2\pi)^{-s} \Gamma(s) \Gamma(s - k + 2) Z_F(s)$  is  $(-1)^k$ -invariant under  $s \mapsto 2k - 2 - s$ .

Let  $M_{2k-2}(\Gamma_1)$  be the space of modular forms of weight  $2k - 2$  on  $\Gamma_1$ . Recall that a Hecke eigenform in  $M_{2k-2}(\Gamma_1)$  is called normalized if its first Fourier coefficient is equal to 1.

**THEOREM 2** (Saito-Kurokawa conjecture; Andrianov [2], Maass [28, 29], Zagier [39]). *Let  $k$  be even and let  $F$  be a non-zero Hecke eigenform in  $M_k^*(\Gamma_2)$ . Then there is a unique normalized Hecke eigenform  $f$  in  $M_{2k-2}(\Gamma_1)$  such that*

$$Z_F(s) = \zeta(s - k + 1) \zeta(s - k + 2) L_f(s)$$

where  $L_f(s)$  is the Hecke  $L$ -function attached to  $f$ .

Theorem 2 in particular shows that  $Z_F(s)$  has a pole at  $s = k$  if  $F$  is a Hecke eigenform in  $S_k^*(\Gamma_2)$ . The converse is also true as shown by Evdokimov [10] and Oda [31], i.e. the function  $Z_F(s)$  is holomorphic everywhere if and only if  $F$  lies in the orthogonal complement of  $S_k^*(\Gamma_2)$ .

Using Theorem 2 one can show that  $M_k^*(\Gamma_2) = \bigcap_p \ker \mathcal{O}_p$  where  $\mathcal{O}_p$  is defined by (5). Finally let us mention that Theorem 2 implies that a Hecke eigenform  $F$  in  $S_k^*(\Gamma_2)$  does not satisfy the generalized Ramanujan-Petersson conjecture which would require that  $\lambda_n \ll_{\varepsilon, F} n^{k-3/2+\varepsilon}$  ( $\varepsilon > 0$ ).

The proof of Theorem 1 is based on the fact that the function  $V\phi$ , by definition, is symmetric w.r.t.  $\tau$  and  $\tau'$  and that  $\Gamma_2$  is generated by the matrix  $\operatorname{diag} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$  (which acts on  $\mathcal{H}_2$  by interchanging  $\tau$  and  $\tau'$ ) and the image of  $\Gamma_1^J$  under the map (2). For the compatibility statement of  $V$  with Hecke operators one has to check the action of the latter on Fourier coeffi-

cients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

## 2.2. PROBLEMS

i) Since for fixed  $k$  the dimension of  $J_{k,m}$  grows linearly in  $m$ , the map  $\rho_m$  defined by (3) for  $m \gg_k 0$  cannot be surjective. Is there any simple or nice description of the image of  $\rho_m$  or  $(im \rho_m | S_k(\Gamma_2))^\perp$ ? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on  $\Gamma_2$  which generate  $S_k(\Gamma_2)$ , as certain infinite linear combinations of Poincaré series on  $\Gamma_1^J$  [22]. Taking scalar products one obtains a characterization of  $(im \rho_m | S_k(\Gamma_2))^\perp$  as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that  $\rho_1$  is surjective).

ii) A skew-holomorphic Jacobi form of weight  $k \in \mathbf{Z}$  and index  $m \in \mathbf{N}_0$  on  $\Gamma_1^J$  as introduced by Skoruppa is a complex-valued  $C^\infty$ -function  $\phi(\tau, z)$  ( $\tau \in \mathcal{H}$ ,  $z \in \mathbf{C}$ ) satisfying the following properties: 1)  $\phi$  is holomorphic in  $z$  and is annihilated by the heat operator  $8\pi im \partial / \partial \tau - \partial^2 / \partial z^2$ ; 2)  $\phi$  satisfies the same transformation formula under  $\Gamma_1^J$  as a holomorphic Jacobi form of weight  $k$  and index  $m$  (cf. § 1.2) except that the factor  $(c\tau + d)^k$  has to be replaced by  $(c\bar{\tau} + d)^{k-1} |c\tau + d|$ ; 3)  $\phi$  has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \geq 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \quad (v = \text{Im}(\tau)).$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of  $F(Z)$  w.r.t.  $e^{2\pi i \tau'}$  (where as usual  $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ ) not only depend on  $\tau$  and  $z$  but also on  $\text{Im}(\tau')$ , and it is a priori not obvious how to get rid of the latter variable and to produce “true” Jacobi forms.

Let  $k$  be an odd integer and denote by  $M_{1/2, k-1/2}(\Gamma_2)$  the space of Siegel-Maass wave forms “of type  $(1/2, k-1/2)$ ” as defined in [26], i.e. the space of real-analytic functions  $F: \mathcal{H}_2 \rightarrow \mathbf{C}$  which satisfy

$$F(M \langle Z \rangle) = \det(C\bar{Z} + D)^{k-1} | \det(CZ + D) | F(Z)$$