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Artikel: JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS
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only on the discriminant $D := r^2 - 4mn$ and the residue class $r \pmod{2m}$.

The Petersson scalar product on $J_{k,m}^{\mathrm{cusp}}$ is normalized by

$$< \phi, \psi > = \int_{\Gamma_1^J \backslash \mathscr{H} \times \mathbf{C}} \phi(\tau, z)\overline{\psi(\tau, z)} \exp(-4\pi m y^2/v) v^{k-3} du dv dx dy$$

$$(\tau = u + iv, \ z = x + iy) .$$

For basic facts about Jacobi forms we refer to [9].

§2. THE MAASS SPACE

2.1. RESULTS

Let F be a Siegel modular form of integral weight k on Γ_2 and write the Fourier expansion of F in the form

(1) $$F(Z) = \sum_{m \geqslant 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \quad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2\right) .$$

Using the injection

(2) $$\Gamma_1^J \to \Gamma_2 , \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the transformation formula of F it is easy to see that the functions ϕ_m are in $J_{k,m}$. The expansion (1) is referred to as the Fourier-Jacobi expansion of F.

Thus for any $m \in \mathbf{N}_0$ we obtain a linear map

(3) $$\rho_m : M_k(\Gamma_2) \to J_{k,m} , \ F \mapsto \phi_m .$$

Note that ρ_0 is equal to the Siegel Φ-operator.

We shall be interested in the case $m = 1$. For k odd, ρ_1 is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For k even, ρ_1 was studied in detail by Maass [28, 29] who showed the existence of a natural map $V: J_{k,1} \to M_k(\Gamma_2)$ such that the composite $\rho_1 \circ V$ is the identity. More precisely, let $\phi \in J_{k,1}$ with Fourier coefficients $c(n, r)$ $(n, r \in \mathbf{Z}; r^2 \leqslant 4n)$ and for $m \in \mathbf{N}_0$ define

(4) $$(V_m\phi)(\tau, z) := \sum_{n,r\in\mathbf{Z},\, r^2\leqslant 4mn} \left(\sum_{d\mid(n,r,m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right)\right) q^n \zeta^r$$

(if $m = 0$, the term $\sum_{d\mid 0} d^{k-1}c(0,0)$ on the right of (4) has to be interpreted as $\frac{1}{2}\zeta(1-k)$; note that $V_1\phi = \phi$). Using a more invariant definition of V_m in terms of the action of a set of representatives for $\Gamma_1 \backslash \{M \in \mathbf{Z}^{(2,2)} \mid \det M = m\}$ one checks that $V_m\phi \in J_{k,m}$ [9, §4]. Put

$$(V\phi)(Z) := \sum_{m\geqslant 0} (V_m\phi)(\tau, z) e^{2\pi i m\tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2\right).$$

We denote by $T_n (n \in \mathbf{N})$ the usual Hecke operators on $M_k(\Gamma_2)$ resp. $S_k(\Gamma_2)$ [12, IV; 1, II]; thus, if p is a prime, T_p resp. T_{p^2} correspond to the two generators

$$\Gamma_2 \begin{pmatrix} 1_2 & 0 \\ 0 & p1_2 \end{pmatrix} \Gamma_2 \text{ resp. } \Gamma_2 \operatorname{diag}(1, p, p^2, p)\Gamma_2$$

of the local Hecke algebra of Γ_2 at p. We denote by $T_{J,n} (n \in \mathbf{N})$ the Hecke operators on $J_{k,m}$ resp. $J_{k,m}^{\text{cusp}}$ [9, §4].

THEOREM 1. (Maass [28, 29], Andrianov [2]). *Suppose that* k *is even. The map* $\phi \mapsto V\phi$ *gives an injection* $J_{k,1} \to M_k(\Gamma_2)$ *which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. If* p *is a prime, one has* $T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$ *and* $T_{p^2} \circ V = V \circ (T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2})$.

The image of $J_{k,1}$ under V is called the Maass space and will be denoted by $M_k^*(\Gamma_2)$. One knows that $M_k^*(\Gamma_2) = \mathbf{C}E_k^{(2)} \oplus S_k^*(\Gamma_2)$ where $E_k^{(2)}$ is the Siegel-Eisenstein series of weight k on Γ_2 and $S_k^*(\Gamma_2) := M_k^*(\Gamma_2) \cap S_k(\Gamma_2)$. Observe that $\dim M_k^*(\Gamma_2) = \dim J_{k,1}$ grows linearly in k while $\dim M_k(\Gamma_2)$ grows like k^3.

Note that Theorem 1 implies that $M_k^*(\Gamma_2)$ is stable under all Hecke operators and that it is annihilated by the operator

(5) $$\mathscr{C}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2},$$

for every prime p.

Let $F \in M_k(\Gamma_2)$ be a non-zero Hecke eigenform and denote by $\lambda_n (n \in \mathbf{N})$ its eigenvalues under T_n. If p is a prime, we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3}X^3 + p^{4k-6}X^4$$

so that $Z_{F,p}\ (p^{-s})$ $(s \in \mathbf{C})$ is the local spinor zeta function of F at p. We put

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s}) \qquad (\mathrm{Re}(s) \gg 0)\,.$$

One has

$$Z_F(s) = \zeta(2s - 2k + 4)^{-1} \sum_{n \geqslant 1} \lambda_n n^{-s} \qquad (\mathrm{Re}(s) \gg 0)\,.$$

If F is an Eisenstein series, then it is well-known that $Z_F(s)$ can be expressed in terms of products of Hecke L-functions of elliptic modular forms.

Suppose that F is cuspidal. Then it was proved in [1, Chap. 3] that $Z_F(s)$ has a meromorphic continuation to $\mathbf{C}$ which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s = k$ if k is even. Moreover, the global function $Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s - k + 2)Z_F(s)$ is $(-1)^k$-invariant under $s \mapsto 2k - 2 - s$.

Let $M_{2k-2}(\Gamma_1)$ be the space of modular forms of weight $2k - 2$ on Γ_1. Recall that a Hecke eigenform in $M_{2k-2}(\Gamma_1)$ is called normalized if its first Fourier coefficient is equal to 1.

THEOREM 2 (Saito-Kurokawa conjecture; Andrianov [2], Maass [28, 29], Zagier [39]). *Let* k *be even and let* F *be a non-zero Hecke eigenform in* $M_k^*(\Gamma_2)$. *Then there is a unique normalized Hecke eigenform* f *in* $M_{2k-2}(\Gamma_1)$ *such that*

$$Z_F(s) = \zeta(s - k + 1)\zeta(s - k + 2)L_f(s)$$

where $L_f(s)$ *is the Hecke L-function attached to* f.

Theorem 2 in particular shows that $Z_F(s)$ has a pole at $s = k$ if F is a Hecke eigenform in $S_k^*(\Gamma_2)$. The converse is also true as shown by Evdokimov [10] and Oda [31], i.e. the function $Z_F(s)$ is holomorphic everywhere if and only if F lies in the orthogonal complement of $S_k^*(\Gamma_2)$.

Using Theorem 2 one can show that $M_k^*(\Gamma_2) = \bigcap_p \ker \mathscr{C}_p$ where $\mathscr{C}_p$ is defined by (5). Finally let us mention that Theorem 2 implies that a Hecke eigenform F in $S_k^*(\Gamma_2)$ does not satisfy the generalized Ramanujan-Petersson conjecture which would require that $\lambda_n \ll_{\varepsilon,F} n^{k-3/2+\varepsilon}$ $(\varepsilon > 0)$.

The proof of Theorem 1 is based on the fact that the function $V\phi$, by definition, is symmetric w.r.t. τ and τ' and that Γ_2 is generated by the matrix $\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ (which acts on $\mathscr{H}_2$ by interchanging τ and τ') and the image of Γ_1^J under the map (2). For the compatibility statement of V with Hecke operators one has to check the action of the latter on Fourier coefficients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

2.2. PROBLEMS

i) Since for fixed k the dimension of $J_{k,m}$ grows linearly in m, the map ρ_m defined by (3) for $m \gg_k 0$ cannot be surjective. Is there any simple or nice description of the image of ρ_m or $(im\,\rho_m \mid S_k(\Gamma_2))^{\perp}$? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on Γ_2 which generate $S_k(\Gamma_2)$, as certain infinite linear combinations of Poincaré series on Γ_1^J [22]. Taking scalar products one obtains a characterization of $(im\,\rho_m \mid S_k(\Gamma_2))^{\perp}$ as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that ρ_1 is surjective).

ii) A skew-holomorphic Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}_0$ on Γ_1^J as introduced by Skoruppa is a complex-valued C^∞-function $\phi(\tau, z)$ $(\tau \in \mathscr{H}, z \in \mathbf{C})$ satisfying the following properties: 1) ϕ is holomorphic in z and is annihilated by the heat operator $8\pi i m \partial/\partial\tau - \partial^2/\partial z^2$; 2) ϕ satisfies the same transformation formula under Γ_1^J as a holomorphic Jacobi form of weight k and index m (cf. § 1.2) except that the factor $(c\tau + d)^k$ has to be replaced by $(c\bar{\tau} + d)^{k-1} \mid c\tau + d \mid$; 3) ϕ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \geqslant 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \qquad (v = \operatorname{Im}(\tau)) .$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of $F(Z)$ w.r.t. $e^{2\pi i \tau'}$ $\left(\text{where as usual } Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right)$ not only depend on τ and z but also on $\operatorname{Im}(\tau')$, and it is a priori not obvious how to get rid of the latter variable and to produce "true" Jacobi forms.

Let k be an odd integer and denote by $M_{1/2, k-1/2}(\Gamma_2)$ the space of Siegel-Maass wave forms "of type $(1/2, k - 1/2)$" as defined in [26], i.e. the space of real-analytic functions $F: \mathscr{H}_2 \to \mathbf{C}$ which satisfy

$$F(M\langle Z\rangle) = \det(C\bar{Z} + D)^{k-1} \mid \det(CZ + D) \mid F(Z)$$