§2. The Maass space

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only on the discriminant $D := r^2 - 4mn$ and the residue class $r(\mathrm{mod}\, 2m)$.

The Petersson scalar product on $J_{k,m}^{\mathrm{cusp}}$ is normalized by

$$<\phi, \psi> = \int_{\Gamma_1^J \backslash \mathscr{H} \times \mathbf{C}} \phi(\tau, z)\overline{\psi(\tau, z)} \exp(-4\pi m y^2/v) v^{k-3} du dv dx dy$$

$$(\tau = u + iv, \; z = x + iy) \; .$$

For basic facts about Jacobi forms we refer to [9].

§2. The Maass space

2.1. Results

Let F be a Siegel modular form of integral weight k on Γ_2 and write the Fourier expansion of F in the form

$$F(Z) = \sum_{m \geqslant 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \quad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2 \right) . \tag{1}$$

Using the injection

$$\Gamma_1^J \to \Gamma_2 \, , \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa) \right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where $(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the transformation formula of F it is easy to see that the functions ϕ_m are in $J_{k,m}$. The expansion (1) is referred to as the Fourier-Jacobi expansion of F.

Thus for any $m \in \mathbf{N}_0$ we obtain a linear map

$$\rho_m : M_k(\Gamma_2) \to J_{k,m} \, , \; F \mapsto \phi_m \, . \tag{3}$$

Note that ρ_0 is equal to the Siegel Φ-operator.

We shall be interested in the case $m = 1$. For k odd, ρ_1 is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For k even, ρ_1 was studied in detail by Maass [28, 29] who showed the existence of a natural map $V: J_{k,1} \to M_k(\Gamma_2)$ such that the composite $\rho_1 \circ V$ is the identity. More precisely, let $\phi \in J_{k,1}$ with Fourier coefficients $c(n, r)$ $(n, r \in \mathbf{Z}; r^2 \leqslant 4n)$ and for $m \in \mathbf{N}_0$ define

(4) $$(V_m\phi)(\tau, z) := \sum_{n,r\in\mathbf{Z},\, r^2\leqslant 4mn} \left(\sum_{d\mid(n,r,m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right)\right) q^n\zeta^r$$

(if $m = 0$, the term $\sum_{d\mid 0} d^{k-1}c(0,0)$ on the right of (4) has to be interpreted as $\frac{1}{2}\zeta(1-k)$; note that $V_1\phi = \phi$). Using a more invariant definition of V_m in terms of the action of a set of representatives for $\Gamma_1 \backslash \{M \in \mathbf{Z}^{(2,2)} \mid \det M = m\}$ one checks that $V_m\phi \in J_{k,m}$ [9, §4]. Put

$$(V\phi)(Z) := \sum_{m\geqslant 0} (V_m\phi)(\tau, z)e^{2\pi i m\tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2\right).$$

We denote by $T_n (n \in \mathbf{N})$ the usual Hecke operators on $M_k(\Gamma_2)$ resp. $S_k(\Gamma_2)$ [12, IV; 1, II]; thus, if p is a prime, T_p resp. T_{p^2} correspond to the two generators

$$\Gamma_2 \begin{pmatrix} 1_2 & 0 \\ 0 & p1_2 \end{pmatrix} \Gamma_2 \text{ resp. } \Gamma_2 \operatorname{diag}(1, p, p^2, p)\Gamma_2$$

of the local Hecke algebra of Γ_2 at p. We denote by $T_{J,n} (n \in \mathbf{N})$ the Hecke operators on $J_{k,m}$ resp. $J_{k,m}^{\text{cusp}}$ [9, §4].

THEOREM 1. (Maass [28, 29], Andrianov [2]). *Suppose that k is even. The map $\phi \mapsto V\phi$ gives an injection $J_{k,1} \to M_k(\Gamma_2)$ which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. If p is a prime, one has $T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$ and $T_{p^2} \circ V = V \circ (T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2})$.*

The image of $J_{k,1}$ under V is called the Maass space and will be denoted by $M_k^*(\Gamma_2)$. One knows that $M_k^*(\Gamma_2) = \mathbf{C}E_k^{(2)} \oplus S_k^*(\Gamma_2)$ where $E_k^{(2)}$ is the Siegel-Eisenstein series of weight k on Γ_2 and $S_k^*(\Gamma_2) := M_k^*(\Gamma_2) \cap S_k(\Gamma_2)$. Observe that $\dim M_k^*(\Gamma_2) = \dim J_{k,1}$ grows linearly in k while $\dim M_k(\Gamma_2)$ grows like k^3.

Note that Theorem 1 implies that $M_k^*(\Gamma_2)$ is stable under all Hecke operators and that it is annihilated by the operator

(5) $$\mathscr{C}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2},$$

for every prime p.

Let $F \in M_k(\Gamma_2)$ be a non-zero Hecke eigenform and denote by $\lambda_n (n \in \mathbf{N})$ its eigenvalues under T_n. If p is a prime, we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3}X^3 + p^{4k-6}X^4$$

so that $Z_{F,p}\ (p^{-s})$ $(s \in \mathbf{C})$ is the local spinor zeta function of F at p. We put

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s}) \qquad (\mathrm{Re}(s) \gg 0) .$$

One has

$$Z_F(s) = \zeta(2s-2k+4)^{-1} \sum_{n \geqslant 1} \lambda_n n^{-s} \qquad (\mathrm{Re}(s) \gg 0) .$$

If F is an Eisenstein series, then it is well-known that $Z_F(s)$ can be expressed in terms of products of Hecke L-functions of elliptic modular forms.

Suppose that F is cuspidal. Then it was proved in [1, Chap. 3] that $Z_F(s)$ has a meromorphic continuation to $\mathbf{C}$ which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s = k$ if k is even. Moreover, the global function $Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$ is $(-1)^k$-invariant under $s \mapsto 2k-2-s$.

Let $M_{2k-2}(\Gamma_1)$ be the space of modular forms of weight $2k-2$ on Γ_1. Recall that a Hecke eigenform in $M_{2k-2}(\Gamma_1)$ is called normalized if its first Fourier coefficient is equal to 1.

THEOREM 2 (Saito-Kurokawa conjecture; Andrianov [2], Maass [28, 29], Zagier [39]). *Let k be even and let F be a non-zero Hecke eigenform in $M_k^*(\Gamma_2)$. Then there is a unique normalized Hecke eigenform f in $M_{2k-2}(\Gamma_1)$ such that*

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s)$$

where $L_f(s)$ is the Hecke L-function attached to f.

Theorem 2 in particular shows that $Z_F(s)$ has a pole at $s = k$ if F is a Hecke eigenform in $S_k^*(\Gamma_2)$. The converse is also true as shown by Evdokimov [10] and Oda [31], i.e. the function $Z_F(s)$ is holomorphic everywhere if and only if F lies in the orthogonal complement of $S_k^*(\Gamma_2)$.

Using Theorem 2 one can show that $M_k^*(\Gamma_2) = \bigcap_p \ker \mathscr{C}_p$ where $\mathscr{C}_p$ is defined by (5). Finally let us mention that Theorem 2 implies that a Hecke eigenform F in $S_k^*(\Gamma_2)$ does not satisfy the generalized Ramanujan-Petersson conjecture which would require that $\lambda_n \ll_{\varepsilon,F} n^{k-3/2+\varepsilon}$ $(\varepsilon > 0)$.

The proof of Theorem 1 is based on the fact that the function $V\phi$, by definition, is symmetric w.r.t. τ and τ' and that Γ_2 is generated by the matrix $\mathrm{diag}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$ (which acts on $\mathscr{H}_2$ by interchanging τ and τ') and the image of Γ_1^J under the map (2). For the compatibility statement of V with Hecke operators one has to check the action of the latter on Fourier coefficients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

2.2. PROBLEMS

i) Since for fixed k the dimension of $J_{k,m}$ grows linearly in m, the map ρ_m defined by (3) for $m \gg_k 0$ cannot be surjective. Is there any simple or nice description of the image of ρ_m or $(im\,\rho_m \mid S_k(\Gamma_2))^{\perp}$? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on Γ_2 which generate $S_k(\Gamma_2)$, as certain infinite linear combinations of Poincaré series on Γ_1^J [22]. Taking scalar products one obtains a characterization of $(im\,\rho_m \mid S_k(\Gamma_2))^{\perp}$ as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that ρ_1 is surjective).

ii) A skew-holomorphic Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}_0$ on Γ_1^J as introduced by Skoruppa is a complex-valued C^∞-function $\phi(\tau, z)$ $(\tau \in \mathscr{H}, z \in \mathbf{C})$ satisfying the following properties: 1) ϕ is holomorphic in z and is annihilated by the heat operator $8\pi i m\partial/\partial\tau - \partial^2/\partial z^2$; 2) ϕ satisfies the same transformation formula under Γ_1^J as a holomorphic Jacobi form of weight k and index m (cf. § 1.2) except that the factor $(c\tau + d)^k$ has to be replaced by $(c\bar{\tau} + d)^{k-1} \mid c\tau + d \mid$; 3) ϕ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \geqslant 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \qquad (v = \mathrm{Im}(\tau)) .$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of $F(Z)$ w.r.t. $e^{2\pi i \tau'}$ $\left(\text{where as usual } Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right)$ not only depend on τ and z but also on $\mathrm{Im}(\tau')$, and it is a priori not obvious how to get rid of the latter variable and to produce "true" Jacobi forms.

Let k be an odd integer and denote by $M_{1/2, k-1/2}(\Gamma_2)$ the space of Siegel-Maass wave forms "of type $(1/2, k - 1/2)$" as defined in [26], i.e. the space of real-analytic functions F: $\mathscr{H}_2 \to \mathbf{C}$ which satisfy

$$F(M\langle Z\rangle) = \det(C\bar{Z} + D)^{k-1} \mid \det(CZ + D) \mid F(Z)$$

for all $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$ and which are annihilated by the matrix differential operator

$$\Omega_{1/2,k-1/2} := (Z-\bar{Z})\left((Z-\bar{Z})\frac{\partial}{\partial Z}\right)'\frac{\partial}{\partial \bar{Z}} + \frac{1}{2}(Z-\bar{Z})\frac{\partial}{\partial \bar{Z}}$$
$$- \left(k-\frac{1}{2}\right)(Z-\bar{Z})\frac{\partial}{\partial Z}$$

$$\text{where } \frac{\partial}{\partial Z} = \begin{pmatrix} \dfrac{\partial}{\partial \tau} & \dfrac{1}{2}\dfrac{\partial}{\partial z} \\ \dfrac{1}{2}\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial \tau'} \end{pmatrix}$$

and $\dfrac{\partial}{\partial \bar{Z}}$ is defined analogously (the notation "of type $(1/2, k-1/2)$" comes from the fact that the factor of automorphy of F can be written as $\det(C\bar{Z}+D)^{k-1/2}\det(CZ+D)^{1/2}$ with appropriate choice of the square root).

Using certain invariance properties of $\Omega_{1/2,k-1/2}$ under the action of $Sp_2(\mathbf{R})$ one can define Hecke operators $T_n (n \in \mathbf{N})$ on $M_{1/2,k-1/2}(\Gamma_2)$ in the usual way. Let

$$E^{(2)}_{1/2,k-1/2}(Z) := \sum_{(C,D)} \det(CZ+D)^{-k+1}\,|\det((CZ+D))|^{-1} \qquad (k>3)$$

be the Maass-Siegel-Eisenstein series in $M_{1/2,k-1/2}(\Gamma_2)$ ([26; 27, § 18]; summation over all pairs (C, D) of relatively prime symmetric (2, 2)-matrices inequivalent under left-multiplication by $GL_2(\mathbf{Z})$). Then the following can be shown:

1) The function $E^{(2)}_{1/2,k-1/2}$ is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to $\zeta(s-k+1)$ $\zeta(s-k+2)L_{E_{2k-2}}(s)$ where E_{2k-2} is the normalized Eisenstein series of weight $2k-2$ on Γ_1 (this implies that $E^{(2)}_{1/2,k-1/2}$ for all primes p is annihilated by the Hecke operator $\mathscr{T}_p$ defined analogously as in (5));

2) if $e_{1/2,k-1/2;m}(\tau, z, \mathrm{Im}(\tau'))$ is the m-th Fourier-Jacobi coefficient of $E^{(2)}_{1/2,k-1/2}$ and if for $m>0$ one carries out a similar limit process as in [19, § 2, Remark ii) after the proof of Thm. 1], i.e. essentially replaces $\mathrm{Im}(\tau')$ by $(\mathrm{Im}(z))^2/\mathrm{Im}(\tau)+\delta$ and lets $\delta \to \infty$, then one obtains a skew-holomorphic Eisenstein series of weight k and index m (in fact, finite linear combinations of such Eisenstein series if m is not squarefree).

The following questions therefore are suggestive:

1) if one starts with an arbitrary $F \in M_{1/2, k-1/2}(\Gamma_2)$, does the above limit process produce skew-holomorphic Jacobi forms of weight k?

2) define $M^*_{1/2, k-1/2}(\Gamma_2)$ as the subspace of $M_{1/2, k-1/2}(\Gamma_2)$ consisting of the intersection of the kernels of the operators $\mathscr{C}_p$ for all primes p. Does there exist a natural map V from skew-holomorphic Jacobi forms of weight k and index 1 to $M^*_{1/2, k-1/2}(\Gamma_2)$ similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on Sp_2. It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus $n > 2$ has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for $n \geqslant 33$ the map which sends a Siegel modular form of weight 16 on $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$ to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for $n = 3$ due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform F of even integral weight k on Γ_3 could be constructed from a pair (f, g) of elliptic Hecke eigenforms of weights (k_1, k_2) equal to $(k, 2k-4)$ or $(k-2, 2k-2)$ such that the (formal) spinor zeta function of F should be equal to $L_f(s - k_2/2) L_f(s - k_2/2 + 1) L_{f \otimes g}(s)$ where $L_{f \otimes g}(s)$ essentially is the Rankin convolution of f and g ([*loc. cit.*, §4]; note that for $n > 2$ the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on Γ_n is not known).

§3. Spinor zeta functions

3.1. Results

Although the Maass space $S_k^*(\Gamma_2)$ as discussed in the previous section is an important subspace of $S_k(\Gamma_2)$ in its own right, one quickly realizes that the "true" Siegel cusp forms on Γ_2 should lie in the orthogonal complement of $S_k^*(\Gamma_2)$ (cf. Theorem 2 in §2 and its discussion). Is is therefore even more