| Zeitschrift: | L'Enseignement Mathématique                                        |
|--------------|--------------------------------------------------------------------|
| Herausgeber: | Commission Internationale de l'Enseignement Mathématique           |
| Band:        | 39 (1993)                                                          |
| Heft:        | 1-2: L'ENSEIGNEMENT MATHÉMATIQUE                                   |
|              |                                                                    |
| Artikel:     | JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS |
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| Kapitel:     | §2. The Maass space                                                |
| DOI:         | https://doi.org/10.5169/seals-60416                                |

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only on the discriminant  $D := r^2 - 4mn$  and the residue class  $r \pmod{2m}$ . The Petersson scalar product on  $J_{k,m}^{\text{cusp}}$  is normalized by

$$<\phi, \psi> = \int_{\Gamma_1^J \setminus \mathscr{H} \times C} \phi(\tau, z) \overline{\psi(\tau, z)} \exp\left(-4\pi m y^2/v\right) v^{k-3} du dv dx dy$$
$$(\tau = u + iv, \ z = x + iy) \ .$$

For basic facts about Jacobi forms we refer to [9].

# §2. The Maass space

## 2.1. RESULTS

Let F be a Siegel modular form of integral weight k on  $\Gamma_2$  and write the Fourier expansion of F in the form

(1) 
$$F(Z) = \sum_{m \ge 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \quad \left( Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2 \right).$$

Using the injection

(2) 
$$\Gamma_1^J \to \Gamma_2$$
,  $\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu), \kappa \right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 

where  $(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and the transformation formula of F it is

easy to see that the functions  $\phi_m$  are in  $J_{k,m}$ . The expansion (1) is referred to as the Fourier-Jacobi expansion of F.

Thus for any  $m \in \mathbf{N}_0$  we obtain a linear map

(3) 
$$\rho_m: M_k(\Gamma_2) \to J_{k,m}, \ F \mapsto \phi_m.$$

Note that  $\rho_0$  is equal to the Siegel  $\Phi$ -operator.

We shall be interested in the case m = 1. For k odd,  $\rho_1$  is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For k even,  $\rho_1$  was studied in detail by Maass [28, 29] who showed the existence of a natural map  $V: J_{k,1} \to M_k(\Gamma_2)$  such that the composite  $\rho_1 \circ V$  is the identity. More precisely, let  $\phi \in J_{k,1}$  with Fourier coefficients c(n, r)  $(n, r \in \mathbb{Z}; r^2 \leq 4n)$  and for  $m \in \mathbb{N}_0$  define

(4) 
$$(V_m \phi)(\tau, z) := \sum_{n, r \in \mathbb{Z}, r^2 \leq 4mn} \left( \sum_{d \mid (n, r, m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right) \right) q^n \zeta^n$$

(if m = 0, the term  $\sum_{d \mid 0} d^{k-1}c(0, 0)$  on the right of (4) has to be interpreted

as  $\frac{1}{2}\zeta(1-k)$ ; note that  $V_1\phi = \phi$ ). Using a more invariant definition of  $V_m$  in terms of the action of a set of representatives for  $\Gamma_1 \setminus \{M \in \mathbb{Z}^{(2,2)} \mid \det M = m\}$  one checks that  $V_m\phi \in J_{k,m}$  [9, §4]. Put

$$(V\phi)(Z) := \sum_{m \ge 0} (V_m\phi)(\tau, z) e^{2\pi i m \tau'} \quad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2 \right).$$

We denote by  $T_n (n \in \mathbb{N})$  the usual Hecke operators on  $M_k(\Gamma_2)$  resp.  $S_k(\Gamma_2)$ [12, IV; 1, II]; thus, if p is a prime,  $T_p$  resp.  $T_{p^2}$  correspond to the two generators

$$\Gamma_2 \begin{pmatrix} 1_2 & 0 \\ 0 & p 1_2 \end{pmatrix} \Gamma_2$$
 resp.  $\Gamma_2$  diag  $(1, p, p^2, p) \Gamma_2$ 

of the local Hecke algebra of  $\Gamma_2$  at p. We denote by  $T_{J,n}$   $(n \in \mathbb{N})$  the Hecke operators on  $J_{k,m}$  resp.  $J_{k,m}^{cusp}$  [9, §4].

THEOREM 1. (Maass [28, 29], Andrianov [2]). Suppose that k is even. The map  $\phi \mapsto V\phi$  gives an injection  $J_{k,1} \to M_k(\Gamma_2)$  which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. If p is a prime, one has  $T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$  and  $T_{p^2} \circ V = V \circ (T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2}).$ 

The image of  $J_{k,1}$  under V is called the Maass space and will be denoted by  $M_k^*(\Gamma_2)$ . One knows that  $M_k^*(\Gamma_2) = \mathbb{C}E_k^{(2)} \oplus S_k^*(\Gamma_2)$  where  $E_k^{(2)}$  is the Siegel-Eisenstein series of weight k on  $\Gamma_2$  and  $S_k^*(\Gamma_2) := M_k^*(\Gamma_2) \cap S_k(\Gamma_2)$ . Observe that dim  $M_k^*(\Gamma_2) = \dim J_{k,1}$  grows linearly in k while dim  $M_k(\Gamma_2)$ grows like  $k^3$ .

Note that Theorem 1 implies that  $M_k^*(\Gamma_2)$  is stable under all Hecke operators and that it is annihilated by the operator

(5) 
$$\mathscr{D}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2},$$

for every prime p.

Let  $F \in M_k(\Gamma_2)$  be a non-zero Hecke eigenform and denote by  $\lambda_n (n \in \mathbb{N})$  its eigenvalues under  $T_n$ . If p is a prime, we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4}) X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4$$

so that  $Z_{F,p}$   $(p^{-s})$   $(s \in \mathbb{C})$  is the local spinor zeta function of F at p. We put

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s}) \quad (\operatorname{Re}(s) \ge 0) .$$

One has

$$Z_F(s) = \zeta (2s - 2k + 4)^{-1} \sum_{n \ge 1} \lambda_n n^{-s} \quad (\operatorname{Re}(s) \ge 0) .$$

If F is an Eisenstein series, then it is well-known that  $Z_F(s)$  can be expressed in terms of products of Hecke L-functions of elliptic modular forms.

Suppose that F is cuspidal. Then it was proved in [1, Chap. 3] that  $Z_F(s)$  has a meromorphic continuation to C which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at s = k if k is even. Moreover, the global function  $Z_F^*(s) := (2\pi)^{-s} \Gamma(s) \Gamma(s - k + 2) Z_F(s)$  is  $(-1)^k$ -invariant under  $s \mapsto 2k - 2 - s$ .

Let  $M_{2k-2}(\Gamma_1)$  be the space of modular forms of weight 2k - 2 on  $\Gamma_1$ . Recall that a Hecke eigenform in  $M_{2k-2}(\Gamma_1)$  is called normalized if its first Fourier coefficient is equal to 1.

THEOREM 2 (Saito-Kurokawa conjecture; Andrianov [2], Maass [28, 29], Zagier [39]). Let k be even and let F be a non-zero Hecke eigenform in  $M_k^*(\Gamma_2)$ . Then there is a unique normalized Hecke eigenform f in  $M_{2k-2}(\Gamma_1)$  such that

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s)$$

where  $L_f(s)$  is the Hecke L-function attached to f.

Theorem 2 in particular shows that  $Z_F(s)$  has a pole at s = k if F is a Hecke eigenform in  $S_k^*(\Gamma_2)$ . The converse is also true as shown by Evdokimov [10] and Oda [31], i.e. the function  $Z_F(s)$  is holomorphic everywhere if and only if F lies in the orthogonal complement of  $S_k^*(\Gamma_2)$ .

Using Theorem 2 one can show that  $M_k^*(\Gamma_2) = \bigcap_p \ker \mathcal{O}_p$  where  $\mathcal{O}_p$  is defined by (5). Finally let us mention that Theorem 2 implies that a Hecke eigenform F in  $S_k^*(\Gamma_2)$  does not satisfy the generalized Ramanujan-Petersson conjecture which would require that  $\lambda_n \ll_{\varepsilon, F} n^{k-3/2+\varepsilon}$  ( $\varepsilon > 0$ ).

The proof of Theorem 1 is based on the fact that the function  $V\phi$ , by definition, is symmetric w.r.t.  $\tau$  and  $\tau'$  and that  $\Gamma_2$  is generated by the matrix diag  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  (which acts on  $\mathscr{H}_2$  by interchanging  $\tau$  and  $\tau'$ ) and the image of  $\Gamma_1^J$  under the map (2). For the compatibility statement of V with Hecke operators one has to check the action of the latter on Fourier coeffi-

cients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

# 2.2. PROBLEMS

i) Since for fixed k the dimension of  $J_{k,m}$  grows linearly in m, the map  $\rho_m$  defined by (3) for  $m \gg_k 0$  cannot be surjective. Is there any simple or nice description of the image of  $\rho_m$  or  $(im \rho_m | S_k(\Gamma_2))^{\perp}$ ? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on  $\Gamma_2$  which generate  $S_k(\Gamma_2)$ , as certain infinite linear combinations of Poincaré series on  $\Gamma_1^J$  [22]. Taking scalar products one obtains a characterization of  $(im \rho_m | S_k(\Gamma_2))^{\perp}$  as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that  $\rho_1$  is surjective).

ii) A skew-holomorphic Jacobi form of weight  $k \in \mathbb{Z}$  and index  $m \in \mathbb{N}_0$ on  $\Gamma_1^J$  as introduced by Skoruppa is a complex-valued  $C^{\infty}$ -function  $\phi(\tau, z)$  ( $\tau \in \mathcal{H}, z \in \mathbb{C}$ ) satisfying the following properties: 1)  $\phi$  is holomorphic in z and is annihilated by the heat operator  $8\pi im\partial/\partial \tau - \partial^2/\partial z^2$ ; 2)  $\phi$ satisfies the same transformation formula under  $\Gamma_1^J$  as a holomorphic Jacobi form of weight k and index m (cf. §1.2) except that the factor  $(c\tau + d)^k$  has to be replaced by  $(c\overline{\tau} + d)^{k-1} | c\tau + d |$ ; 3)  $\phi$  has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbb{Z}, r^2 \ge 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \quad (v = \operatorname{Im}(\tau)) .$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of F(Z) w.r.t.  $e^{2\pi i \tau'}$  (where as usual  $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ ) not only depend on  $\tau$  and z but also on  $\text{Im}(\tau')$ , and it is a priori not obvious how to get rid of the latter variable and to produce "true" Jacobi forms.

Let k be an odd integer and denote by  $M_{1/2, k-1/2}(\Gamma_2)$  the space of Siegel-Maass wave forms "of type (1/2, k - 1/2)" as defined in [26], i.e. the space of real-analytic functions  $F: \mathscr{H}_2 \to \mathbb{C}$  which satisfy

$$F(M < Z >) = \det(CZ + D)^{k-1} |\det(CZ + D)| F(Z)$$

for all  $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$  and which are annihilated by the matrix differential operator

$$\Omega_{1/2, k-1/2} := (Z - \bar{Z}) \left( (Z - \bar{Z}) \frac{\partial}{\partial Z} \right)' \frac{\partial}{\partial \bar{Z}} + \frac{1}{2} (Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}}$$
$$- \left( k - \frac{1}{2} \right) (Z - \bar{Z}) \frac{\partial}{\partial Z}$$
where  $\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial \tau} & \frac{1}{2} \frac{\partial}{\partial z} \\ \frac{1}{2} \frac{\partial}{\partial z} & \frac{\partial}{\partial \tau'} \end{pmatrix}$ 

and  $\frac{\sigma}{\partial \overline{Z}}$  is defined analogously (the notation "of type (1/2, k - 1/2)" comes from the fact that the factor of automorphy of F can be written as

det  $(C\overline{Z} + D)^{k-1/2}$  det  $(CZ + D)^{1/2}$  with appropriate choice of the square root).

Using certain invariance properties of  $\Omega_{1/2, k-1/2}$  under the action of  $\operatorname{Sp}_2(\mathbf{R})$  one can define Hecke operators  $T_n (n \in \mathbf{N})$  on  $M_{1/2, k-1/2}(\Gamma_2)$  in the usual way. Let

$$E_{1/2,k-1/2}^{(2)}(Z) := \sum_{(C,D)} \det(CZ+D)^{-k+1} \left| \det((CZ+D)) \right|^{-1} \quad (k>3)$$

be the Maass-Siegel-Eisenstein series in  $M_{1/2, k-1/2}(\Gamma_2)$  ([26; 27, §18]; summation over all pairs (*C*, *D*) of relatively prime symmetric (2, 2)-matrices inequivalent under left-multiplication by  $GL_2(\mathbb{Z})$ ). Then the following can be shown:

1) The function  $E_{1/2, k-1/2}^{(2)}$  is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to  $\zeta(s-k+1)$  $\zeta(s-k+2)L_{E_{2k-2}}(s)$  where  $E_{2k-2}$  is the normalized Eisenstein series of weight 2k - 2 on  $\Gamma_1$  (this implies that  $E_{1/2, k-1/2}^{(2)}$  for all primes p is annihilated by the Hecke operator  $\mathcal{O}_p$  defined analogously as in (5));

2) if  $e_{1/2, k-1/2; m}(\tau, z, \operatorname{Im}(\tau'))$  is the *m*-th Fourier-Jacobi coefficient of  $E_{1/2, k-1/2}^{(2)}$  and if for m > 0 one carries out a similar limit process as in [19, §2, Remark ii) after the proof of Thm. 1], i.e. essentially replaces  $\operatorname{Im}(\tau')$  by  $(\operatorname{Im}(z))^2/\operatorname{Im}(\tau) + \delta$  and lets  $\delta \to \infty$ , then one obtains a skew-holomorphic Eisenstein series of weight k and index m (in fact, finite linear combinations of such Eisenstein series if m is not squarefree).

The following questions therefore are suggestive:

1) if one starts with an arbitrary  $F \in M_{1/2, k-1/2}(\Gamma_2)$ , does the above limit process produce skew-holomorphic Jacobi forms of weight k?

2) define  $M_{1/2, k-1/2}^*(\Gamma_2)$  as the subspace of  $M_{1/2, k-1/2}(\Gamma_2)$  consisting of the intersection of the kernels of the operators  $\mathcal{O}_p$  for all primes p. Does there exist a natural map V from skew-holomorphic Jacobi forms of weight k and index 1 to  $M_{1/2, k-1/2}^*(\Gamma_2)$  similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on  $Sp_2$ . It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus n > 2 has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for  $n \ge 33$  the map which sends a Siegel modular form of weight 16 on  $\Gamma_n := \text{Sp}_n(\mathbb{Z})$  to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for n = 3 due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform F of even integral weight k on  $\Gamma_3$  could be constructed from a pair (f, g) of elliptic Hecke eigenforms of weights  $(k_1, k_2)$  equal to (k, 2k - 4) or (k - 2, 2k - 2) such that the (formal) spinor zeta function of F should be equal to  $L_f(s - k_2/2)L_f(s - k_2/2 + 1)L_{f\otimes g}(s)$  where  $L_{f\otimes g}(s)$  essentially is the Rankin convolution of f and g ([loc. cit., §4]; note that for n > 2 the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on  $\Gamma_n$  is not known).

### §3. SPINOR ZETA FUNCTIONS

# 3.1. RESULTS

Although the Maass space  $S_k^*(\Gamma_2)$  as discussed in the previous section is an important subspace of  $S_k(\Gamma_2)$  in its own right, one quickly realizes that the "true" Siegel cusp forms on  $\Gamma_2$  should lie in the orthogonal complement of  $S_k^*(\Gamma_2)$  (cf. Theorem 2 in §2 and its discussion). Is is therefore even more