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Artikel: JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS

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tions $F: \mathscr{H}_2 \to \mathbf{C}$ satisfying $F(M<Z>) = \det(CZ+D)^k F(Z)$ for all $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$. Such a function has a Fourier expansion

$$F(Z) = \sum_{T = T' \geqslant 0} a(T) e^{2\pi i \operatorname{tr}(TZ)}$$

where T runs over all positive semi-definite half-integral (2, 2)-matrices. We write $S_k(\Gamma_2)$ for the subspace of cusp forms (require $a(T) = 0$ for $T \not> 0$).

For $F, G \in S_k(\Gamma_2)$ we denote by

$$<F, G> = \int_{\Gamma_2 \setminus \mathscr{H}_2} F(Z) \overline{G(Z)} (\det Y)^{k-3} dX dY \quad (X = \operatorname{Re}(Z),\ Y = \operatorname{Im}(Z))$$

the Petersson scalar product of F and G.

For basic facts on Siegel modular forms we refer to [12, 17].

1.2. Jacobi forms

We write $\mathscr{H}$ for the complex upper half-plane. We let $H(\mathbf{R})$ be the Heisenberg group, i.e. the set of triples $((\lambda, \mu), \kappa) \in \mathbf{R}^2 \times \mathbf{R}$ with group law $((\lambda, \mu), \kappa)((\lambda', \mu'), \kappa') = ((\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\mu' - \lambda'\mu)$, and denote by $G^J := SL_2(\mathbf{R}) \times H(\mathbf{R})$ the Jacobi group where $SL_2(\mathbf{R})$ operates on $H(\mathbf{R})$ from the right by $((\lambda, \mu), \kappa)M = ((\lambda, \mu)M, \kappa)$. The group G^J acts on $\mathscr{H} \times \mathbf{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

We set $\Gamma_1 := SL_2(\mathbf{Z})$, $\Gamma_1^J := \Gamma_1 \times H(\mathbf{Z})$ and for $k \in \mathbf{Z}$ and $m \in \mathbf{N}_0$ denote by $J_{k,m}$ the space of Jacobi forms of weight k and index m on Γ_1^J, i.e. the space of holomorphic functions $\phi: \mathscr{H} \times \mathbf{C} \to \mathbf{C}$ satisfying the transformation formula

$$\phi(\gamma \circ (\tau, z)) = (c\tau + d)^k \exp\left(2\pi i m \left(\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z\right)\right) \phi(\tau, z)$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \in \Gamma_1^J$ and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z},\, r^2 \leqslant 4mn} c(n, r) q^n \zeta^r$$

where $q = e^{2\pi i \tau}$, $\zeta = e^{2\pi i z}$. We write $J_{k,m}^{\text{cusp}}$ for the subspace of cusp forms (require $c(n, r) = 0$ for $r^2 = 4mn$). Note that the coefficients $c(n, r)$ depend only on the discriminant $D := r^2 - 4mn$ and the residue class $r (\mathrm{mod}\, 2m)$.

The Petersson scalar product on $J_{k,m}^{\mathrm{cusp}}$ is normalized by

$$<\phi, \psi> = \int_{\Gamma_1^J \backslash \mathcal{H} \times \mathbf{C}} \phi(\tau, z)\overline{\psi(\tau, z)} \exp(-4\pi m y^2/v) v^{k-3} du dv dx dy$$

$$(\tau = u + iv\,,\ z = x + iy)\ .$$

For basic facts about Jacobi forms we refer to [9].

§2. The Maass space

2.1. Results

Let F be a Siegel modular form of integral weight k on Γ_2 and write the Fourier expansion of F in the form

(1) $$F(Z) = \sum_{m \geqslant 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2\right).$$

Using the injection

(2) $$\Gamma_1^J \to \Gamma_2\,, \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(\lambda', \mu') = (\lambda, \mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the transformation formula of F it is easy to see that the functions ϕ_m are in $J_{k,m}$. The expansion (1) is referred to as the Fourier-Jacobi expansion of F.

Thus for any $m \in \mathbf{N}_0$ we obtain a linear map

(3) $$\rho_m : M_k(\Gamma_2) \to J_{k,m}\,, \quad F \mapsto \phi_m\,.$$

Note that ρ_0 is equal to the Siegel Φ-operator.

We shall be interested in the case $m = 1$. For k odd, ρ_1 is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For k even, ρ_1 was studied in detail by Maass [28, 29] who showed the existence of a natural map $V: J_{k,1} \to M_k(\Gamma_2)$ such that the composite $\rho_1 \circ V$ is the identity. More precisely, let $\phi \in J_{k,1}$ with Fourier coefficients $c(n, r)$ $(n, r \in \mathbf{Z}; r^2 \leqslant 4n)$ and for $m \in \mathbf{N}_0$ define