JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS

L'Enseignement Mathématique, t. 39 (1993), p. 121-136

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INTRODUCTION

In the present paper we would like to describe some recent developments how Jacobi forms can be used to study Siegel modular forms of genus 2 and what problems arise in this way. After a preliminary section on Siegel modular forms and Jacobi forms (§ 1) which mainly serves to fix some notation, we shall discuss the so called Maass space in § 2. We shall then study relations between Jacobi forms and spinor zeta functions of Hecke eigenforms of genus 2 (§ 3) and finally in § 4 will indicate how Jacobi forms can be used to give estimates for Fourier coefficients of Siegel cusp forms.

Sections 2-4 are divided into two parts: part one describes known results while part two gives some open problems.

We do not go here into any more intrinsic properties of Jacobi forms (as e.g. the trace formula or relations to modular forms of integral weight) nor discuss any representation-theoretic aspects of the theory. For good surveys, we refer to [33, 36] for the first and to [3] for the second topic.

§ 1. PRELIMINARIES ON SIEGEL MODULAR FORMS AND JACOBI FORMS

1.1. SIEGEL MODULAR FORMS OF GENUS 2

We write $\mathcal{H}_2$ for the Siegel upper half-space of genus 2. The natural action of $\mathrm{Sp}_2(\mathbf{R})$ on $\mathcal{H}_2$ is denoted by

$$(M, Z) \mapsto M < Z > \, := (AZ+B)\,(CZ+D)^{-1}$$

$$\left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbf{R})\,, \quad Z \in \mathcal{H}_2\right).$$

We put $\Gamma_2 := \mathrm{Sp}_2(\mathbf{Z})$ and for $k \in \mathbf{Z}$ denote by $M_k(\Gamma_2)$ the space of Siegel modular forms of weight k on Γ_2, i.e. the space of holomorphic functions $F: \mathscr{H}_2 \to \mathbf{C}$ satisfying $F(M<Z>) = \det(CZ+D)^k F(Z)$ for all $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$. Such a function has a Fourier expansion

$$F(Z) = \sum_{T = T' \geqslant 0} a(T) e^{2\pi \operatorname{itr}(TZ)}$$

where T runs over all positive semi-definite half-integral (2, 2)-matrices. We write $S_k(\Gamma_2)$ for the subspace of cusp forms (require $a(T) = 0$ for $T \not> 0$).

For $F, G \in S_k(\Gamma_2)$ we denote by

$$<F, G> = \int_{\Gamma_2 \backslash \mathscr{H}_2} F(Z)\overline{G(Z)} (\det Y)^{k-3} dXdY \qquad (X = \operatorname{Re}(Z),\ Y = \operatorname{Im}(Z))$$

the Petersson scalar product of F and G.

For basic facts on Siegel modular forms we refer to [12, 17].

1.2. Jacobi forms

We write $\mathscr{H}$ for the complex upper half-plane. We let $H(\mathbf{R})$ be the Heisenberg group, i.e. the set of triples $((\lambda, \mu), \kappa) \in \mathbf{R}^2 \times \mathbf{R}$ with group law $((\lambda, \mu), \kappa)((\lambda', \mu'), \kappa') = ((\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\mu' - \lambda'\mu)$, and denote by $G^J := SL_2(\mathbf{R}) \times H(\mathbf{R})$ the Jacobi group where $SL_2(\mathbf{R})$ operates on $H(\mathbf{R})$ from the right by $((\lambda, \mu), \kappa)M = ((\lambda, \mu)M, \kappa)$. The group G^J acts on $\mathscr{H} \times \mathbf{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left((\lambda, \mu), \kappa\right)\right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

We set $\Gamma_1 := SL_2(\mathbf{Z})$, $\Gamma_1^J := \Gamma_1 \times H(\mathbf{Z})$ and for $k \in \mathbf{Z}$ and $m \in \mathbf{N}_0$ denote by $J_{k,m}$ the space of Jacobi forms of weight k and index m on Γ_1^J, i.e. the space of holomorphic functions $\phi: \mathscr{H} \times \mathbf{C} \to \mathbf{C}$ satisfying the transformation formula

$$\phi(\gamma \circ (\tau, z)) = (c\tau + d)^k \exp\left(2\pi i m \left(\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z\right)\right) \phi(\tau, z)$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left((\lambda, \mu), \kappa\right)\right) \in \Gamma_1^J$ and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \leqslant 4mn} c(n, r) q^n \zeta^r$$

where $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$. We write $J_{k,m}^{\text{cusp}}$ for the subspace of cusp forms (require $c(n, r) = 0$ for $r^2 = 4mn$). Note that the coefficients $c(n, r)$ depend only on the discriminant $D := r^2 - 4mn$ and the residue class $r \pmod{2m}$.

The Petersson scalar product on $J_{k,m}^{\text{cusp}}$ is normalized by

$$<\phi, \psi> = \int_{\Gamma_1^J \backslash \mathscr{H} \times \mathbf{C}} \phi(\tau, z)\overline{\psi(\tau, z)} \exp(-4\pi m y^2/v) v^{k-3} du dv dx dy$$

$$(\tau = u + iv\,, \; z = x + iy)\;.$$

For basic facts about Jacobi forms we refer to [9].

§2. THE MAASS SPACE

2.1. RESULTS

Let F be a Siegel modular form of integral weight k on Γ_2 and write the Fourier expansion of F in the form

$$(1) \qquad F(Z) = \sum_{m \geqslant 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2\right).$$

Using the injection

$$(2) \qquad \Gamma_1^J \to \Gamma_2\,, \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the transformation formula of F it is easy to see that the functions ϕ_m are in $J_{k,m}$. The expansion (1) is referred to as the Fourier-Jacobi expansion of F.

Thus for any $m \in \mathbf{N}_0$ we obtain a linear map

$$(3) \qquad \rho_m : M_k(\Gamma_2) \to J_{k,m}\,, \quad F \mapsto \phi_m\,.$$

Note that ρ_0 is equal to the Siegel Φ-operator.

We shall be interested in the case $m = 1$. For k odd, ρ_1 is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For k even, ρ_1 was studied in detail by Maass [28, 29] who showed the existence of a natural map $V: J_{k,1} \to M_k(\Gamma_2)$ such that the composite $\rho_1 \circ V$ is the identity. More precisely, let $\phi \in J_{k,1}$ with Fourier coefficients $c(n, r)$ $(n, r \in \mathbf{Z}; r^2 \leqslant 4n)$ and for $m \in \mathbf{N}_0$ define

(4) $$(V_m\phi)(\tau, z) := \sum_{n, r \in \mathbf{Z}, r^2 \leqslant 4mn} \left(\sum_{d \mid (n, r, m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right)\right) q^n \zeta^r$$

(if $m = 0$, the term $\sum_{d \mid 0} d^{k-1} c(0, 0)$ on the right of (4) has to be interpreted as $\frac{1}{2}\zeta(1-k)$; note that $V_1\phi = \phi$). Using a more invariant definition of V_m in terms of the action of a set of representatives for $\Gamma_1 \backslash \{M \in \mathbf{Z}^{(2,2)} \mid \det M = m\}$ one checks that $V_m\phi \in J_{k,m}$ [9, §4]. Put

$$(V\phi)(Z) := \sum_{m \geqslant 0} (V_m\phi)(\tau, z) e^{2\pi i m \tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2\right).$$

We denote by $T_n (n \in \mathbf{N})$ the usual Hecke operators on $M_k(\Gamma_2)$ resp. $S_k(\Gamma_2)$ [12, IV; 1, II]; thus, if p is a prime, T_p resp. T_{p^2} correspond to the two generators

$$\Gamma_2 \begin{pmatrix} 1_2 & 0 \\ 0 & p1_2 \end{pmatrix} \Gamma_2 \text{ resp. } \Gamma_2 \operatorname{diag}(1, p, p^2, p)\Gamma_2$$

of the local Hecke algebra of Γ_2 at p. We denote by $T_{J,n} (n \in \mathbf{N})$ the Hecke operators on $J_{k,m}$ resp. $J_{k,m}^{\text{cusp}}$ [9, §4].

THEOREM 1. (Maass [28, 29], Andrianov [2]). *Suppose that k is even. The map $\phi \mapsto V\phi$ gives an injection $J_{k,1} \to M_k(\Gamma_2)$ which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. If p is a prime, one has $T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$ and $T_{p^2} \circ V = V \circ (T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2})$.*

The image of $J_{k,1}$ under V is called the Maass space and will be denoted by $M_k^*(\Gamma_2)$. One knows that $M_k^*(\Gamma_2) = \mathbf{C}E_k^{(2)} \oplus S_k^*(\Gamma_2)$ where $E_k^{(2)}$ is the Siegel-Eisenstein series of weight k on Γ_2 and $S_k^*(\Gamma_2) := M_k^*(\Gamma_2) \cap S_k(\Gamma_2)$. Observe that $\dim M_k^*(\Gamma_2) = \dim J_{k,1}$ grows linearly in k while $\dim M_k(\Gamma_2)$ grows like k^3.

Note that Theorem 1 implies that $M_k^*(\Gamma_2)$ is stable under all Hecke operators and that it is annihilated by the operator

(5) $$\mathscr{C}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2},$$

for every prime p.

Let $F \in M_k(\Gamma_2)$ be a non-zero Hecke eigenform and denote by $\lambda_n (n \in \mathbf{N})$ its eigenvalues under T_n. If p is a prime, we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4$$

so that $Z_{F,p}\ (p^{-s})$ $(s \in \mathbf{C})$ is the local spinor zeta function of F at p. We put

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s}) \qquad (\mathrm{Re}(s) \gg 0)\ .$$

One has

$$Z_F(s) = \zeta(2s-2k+4)^{-1} \sum_{n \geqslant 1} \lambda_n n^{-s} \qquad (\mathrm{Re}(s) \gg 0)\ .$$

If F is an Eisenstein series, then it is well-known that $Z_F(s)$ can be expressed in terms of products of Hecke L-functions of elliptic modular forms.

Suppose that F is cuspidal. Then it was proved in [1, Chap. 3] that $Z_F(s)$ has a meromorphic continuation to $\mathbf{C}$ which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s = k$ if k is even. Moreover, the global function $Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$ is $(-1)^k$-invariant under $s \mapsto 2k-2-s$.

Let $M_{2k-2}(\Gamma_1)$ be the space of modular forms of weight $2k-2$ on Γ_1. Recall that a Hecke eigenform in $M_{2k-2}(\Gamma_1)$ is called normalized if its first Fourier coefficient is equal to 1.

THEOREM 2 (Saito-Kurokawa conjecture; Andrianov [2], Maass [28, 29], Zagier [39]). *Let k be even and let F be a non-zero Hecke eigenform in $M_k^*(\Gamma_2)$. Then there is a unique normalized Hecke eigenform f in $M_{2k-2}(\Gamma_1)$ such that*

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s)$$

where $L_f(s)$ is the Hecke L-function attached to f.

Theorem 2 in particular shows that $Z_F(s)$ has a pole at $s = k$ if F is a Hecke eigenform in $S_k^*(\Gamma_2)$. The converse is also true as shown by Evdokimov [10] and Oda [31], i.e. the function $Z_F(s)$ is holomorphic everywhere if and only if F lies in the orthogonal complement of $S_k^*(\Gamma_2)$.

Using Theorem 2 one can show that $M_k^*(\Gamma_2) = \bigcap_p \ker \mathscr{C}_p$ where $\mathscr{C}_p$ is defined by (5). Finally let us mention that Theorem 2 implies that a Hecke eigenform F in $S_k^*(\Gamma_2)$ does not satisfy the generalized Ramanujan-Petersson conjecture which would require that $\lambda_n \ll_{\varepsilon,F} n^{k-3/2+\varepsilon}$ $(\varepsilon > 0)$.

The proof of Theorem 1 is based on the fact that the function $V\phi$, by definition, is symmetric w.r.t. τ and τ' and that Γ_2 is generated by the matrix $\mathrm{diag}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$ (which acts on $\mathscr{H}_2$ by interchanging τ and τ') and the image of Γ_1^J under the map (2). For the compatibility statement of V with Hecke operators one has to check the action of the latter on Fourier coefficients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

2.2. Problems

i) Since for fixed k the dimension of $J_{k,m}$ grows linearly in m, the map ρ_m defined by (3) for $m \gg_k 0$ cannot be surjective. Is there any simple or nice description of the image of ρ_m or $(im\, \rho_m \mid S_k(\Gamma_2))^{\perp}$? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on Γ_2 which generate $S_k(\Gamma_2)$, as certain infinite linear combinations of Poincaré series on Γ_1^J [22]. Taking scalar products one obtains a characterization of $(im\, \rho_m \mid S_k(\Gamma_2))^{\perp}$ as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that ρ_1 is surjective).

ii) A skew-holomorphic Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}_0$ on Γ_1^J as introduced by Skoruppa is a complex-valued C^∞-function $\phi(\tau, z)$ ($\tau \in \mathscr{H}$, $z \in \mathbf{C}$) satisfying the following properties: 1) ϕ is holomorphic in z and is annihilated by the heat operator $8\pi i m \partial/\partial\tau - \partial^2/\partial z^2$; 2) ϕ satisfies the same transformation formula under Γ_1^J as a holomorphic Jacobi form of weight k and index m (cf. § 1.2) except that the factor $(c\tau + d)^k$ has to be replaced by $(c\bar{\tau} + d)^{k-1} \mid c\tau + d \mid$; 3) ϕ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \geqslant 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \qquad (v = \mathrm{Im}(\tau)) .$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of $F(Z)$ w.r.t. $e^{2\pi i \tau'}$ $\left(\text{where as usual } Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right)$ not only depend on τ and z but also on $\mathrm{Im}(\tau')$, and it is a priori not obvious how to get rid of the latter variable and to produce "true" Jacobi forms.

Let k be an odd integer and denote by $M_{1/2, k-1/2}(\Gamma_2)$ the space of Siegel-Maass wave forms "of type $(1/2, k - 1/2)$" as defined in [26], i.e. the space of real-analytic functions F: $\mathscr{H}_2 \to \mathbf{C}$ which satisfy

$$F(M<Z>) = \det(C\bar{Z} + D)^{k-1} \mid \det(CZ + D) \mid F(Z)$$

for all $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$ and which are annihilated by the matrix differential operator

$$\Omega_{1/2, k-1/2} := (Z - \bar{Z}) \left((Z - \bar{Z}) \frac{\partial}{\partial Z} \right)' \frac{\partial}{\partial \bar{Z}} + \frac{1}{2} (Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}} - \left(k - \frac{1}{2} \right) (Z - \bar{Z}) \frac{\partial}{\partial Z}$$

$$\text{where} \quad \frac{\partial}{\partial Z} = \begin{pmatrix} \dfrac{\partial}{\partial \tau} & \dfrac{1}{2} \dfrac{\partial}{\partial z} \\ \dfrac{1}{2} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial \tau'} \end{pmatrix}$$

and $\dfrac{\partial}{\partial \bar{Z}}$ is defined analogously (the notation "of type $(1/2, k - 1/2)$" comes from the fact that the factor of automorphy of F can be written as $\det(C\bar{Z} + D)^{k-1/2} \det(CZ + D)^{1/2}$ with appropriate choice of the square root).

Using certain invariance properties of $\Omega_{1/2, k-1/2}$ under the action of $Sp_2(\mathbf{R})$ one can define Hecke operators $T_n (n \in \mathbf{N})$ on $M_{1/2, k-1/2}(\Gamma_2)$ in the usual way. Let

$$E^{(2)}_{1/2, k-1/2}(Z) := \sum_{(C, D)} \det(CZ + D)^{-k+1} \left| \det((CZ + D)) \right|^{-1} \qquad (k > 3)$$

be the Maass-Siegel-Eisenstein series in $M_{1/2, k-1/2}(\Gamma_2)$ ([26; 27, § 18]; summation over all pairs (C, D) of relatively prime symmetric (2, 2)-matrices inequivalent under left-multiplication by $GL_2(\mathbf{Z})$). Then the following can be shown:

1) The function $E^{(2)}_{1/2, k-1/2}$ is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to $\zeta(s - k + 1) \zeta(s - k + 2) L_{E_{2k-2}}(s)$ where E_{2k-2} is the normalized Eisenstein series of weight $2k - 2$ on Γ_1 (this implies that $E^{(2)}_{1/2, k-1/2}$ for all primes p is annihilated by the Hecke operator $\mathcal{C}_p$ defined analogously as in (5));

2) if $e_{1/2, k-1/2; m}(\tau, z, \mathrm{Im}(\tau'))$ is the m-th Fourier-Jacobi coefficient of $E^{(2)}_{1/2, k-1/2}$ and if for $m > 0$ one carries out a similar limit process as in [19, § 2, Remark ii) after the proof of Thm. 1], i.e. essentially replaces $\mathrm{Im}(\tau')$ by $(\mathrm{Im}(z))^2/\mathrm{Im}(\tau) + \delta$ and lets $\delta \to \infty$, then one obtains a skew-holomorphic Eisenstein series of weight k and index m (in fact, finite linear combinations of such Eisenstein series if m is not squarefree).

The following questions therefore are suggestive:

1) if one starts with an arbitrary $F \in M_{1/2, k-1/2}(\Gamma_2)$, does the above limit process produce skew-holomorphic Jacobi forms of weight k?

2) define $M^*_{1/2, k-1/2}(\Gamma_2)$ as the subspace of $M_{1/2, k-1/2}(\Gamma_2)$ consisting of the intersection of the kernels of the operators $\mathscr{C}_p$ for all primes p. Does there exist a natural map V from skew-holomorphic Jacobi forms of weight k and index 1 to $M^*_{1/2, k-1/2}(\Gamma_2)$ similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on Sp_2. It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus $n > 2$ has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for $n \geqslant 33$ the map which sends a Siegel modular form of weight 16 on $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$ to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for $n = 3$ due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform F of even integral weight k on Γ_3 could be constructed from a pair (f, g) of elliptic Hecke eigenforms of weights (k_1, k_2) equal to $(k, 2k-4)$ or $(k-2, 2k-2)$ such that the (formal) spinor zeta function of F should be equal to $L_f(s - k_2/2) L_f(s - k_2/2 + 1) L_{f \otimes g}(s)$ where $L_{f \otimes g}(s)$ essentially is the Rankin convolution of f and g ([*loc. cit.*, §4]; note that for $n > 2$ the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on Γ_n is not known).

§3. Spinor zeta functions

3.1. Results

Although the Maass space $S^*_k(\Gamma_2)$ as discussed in the previous section is an important subspace of $S_k(\Gamma_2)$ in its own right, one quickly realizes that the "true" Siegel cusp forms on Γ_2 should lie in the orthogonal complement of $S^*_k(\Gamma_2)$ (cf. Theorem 2 in §2 and its discussion). Is is therefore even more surprising that forms in the Maass space can be used to study forms in $S_k^*(\Gamma_2)^\perp$ (in fact, spinor zeta functions of Hecke eigenforms in $S_k^*(\Gamma_2)^\perp$). Thus the importance of the Maass space seems to go much beyond that what is expected from §2.

Let F and G be Siegel cusp forms of integral weight k on Γ_2. Denote by ϕ_m and $\psi_m (m \geqslant 1)$ the Fourier-Jacobi coefficients of F and G, respectively and define a formal Dirichlet series of Rankin-type by

$$D_{F,G}(s) := \zeta(2s-2k+4) \sum_{m \geqslant 1} < \phi_m, \psi_m > m^{-s} \tag{6}$$

(this series was introduced by Skoruppa and the author in [18]).

A variant of the classical Hecke argument shows that $< \phi_m, \psi_m > \ll_{F,G} m^k$ so that $D_{F,G}(s)$ is absolutely convergent for $\mathrm{Re}(s) > k+1$. We put

$$D_{F,G}^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)D_{F,G}(s) \quad (\mathrm{Re}(s) > k+1)\ .$$

THEOREM 1 [18]. *The function* $D_{F,G}(s)$ *has a meromorphic continuation to* $\mathbf{C}$ *which is holomorphic except for a possible simple pole of residue*

$$\frac{4^k \pi^{k+2}}{(k-2)!} < F, G >$$

at $s = k$. *Furthermore, the functional equation*

$$D_{F,G}^*(2k-2-s) = D_{F,G}^*(s)$$

holds.

THEOREM 2 [18]. *Let* k *be even. Let* $F \in S_k(\Gamma_2)$ *be a Hecke eigenform and* G *be a function in the Maass space* $S_k^*(\Gamma_2)$. *Then*

$$D_{F,G}(s) = < \phi_1, \psi_1 > Z_F(s)\ .$$

The proof of Theorem 1 is based on the Rankin-Selberg method applied with an Eisenstein series of Klingen-type on Sp_2. The proof of Theorem 2 uses Theorem 1 of §2 applied with ϕ a Poincaré series; furthermore, an explicit formula for the action on Fourier coefficients of the operator V_m^* adjoint to V_m w.r.t the Petersson scalar products and the relations due to Andrianov [1, Chap. 2] between eigenvalues and Fourier coefficients of Hecke eigenforms play an important role. Let us mention that Theorem 2 could also be deduced from results of Gritsenko [13, p. 266].

In [38], Yamazaki using the theory of Eisenstein series à la Langlands studied the analytic properties of generalizations to arbitrary genus n of the series (6). Recently, Krieg [24] gave a more elementary proof of (some of) the results of [38] using well-known properties of Epstein zeta functions. However, it is clear from the Γ-factors and the type of the functional equations that for $n > 2$ there cannot be any direct connection between the series studied in [24, 38] and spinor zeta functions.

1.2 PROBLEMS

i) Suppose that k is even. If F is a non-zero Hecke eigenform in $S_k(\Gamma_2)$, is $\phi_1 \neq 0$? (This question was already asked in [33].) The answer is positive for $k \leqslant 32$ as numerical computations due to Skoruppa [35] show. Note that by Theorem 2 a positive answer gives a new proof for the analytic continuation and the functional equation of $Z_F(s)$.

ii) Let F be a Hecke eigenform in $S_k(\Gamma_2)$. The only critical point of $Z_F(s)$ in Deligne's sense is $s = k - 1$, i.e. the center of symmetry of the functional equation as is easily checked. Conjecturally therefore $Z_F(k-1)$ should be equal to the determinant of a "period matrix" times an algebraic number (one may suppose that k is even since otherwise $Z_F(k-1) = 0$ as follows from the sign in the functional equation). To the author's knowledge, nothing so far in this direction has been proved. Could Theorem 2 eventually be useful in this context?

As a side remark, let us mention here that Böcherer [4] motivated by Waldspurger's results [37] about the central critical values of quadratic twists of Hecke L-functions of elliptic Hecke eigenforms, for k even has conjectured that the central critical value of the twist of $Z_F(s)$ by a quadratic Dirichlet character of conductor $D < 0$ should be proportional to the *square* of $\sum_{\{T>0\}/\sim,\, \mathrm{disc}\, T = D} a(T)$ where $a(T)$ are the Fourier coefficients of F and the sum is over a set of Γ_1-representatives of positive definite integral binary quadratic forms T of discriminant D. This conjecture is true if F is in the Maass space as follows from Theorem 2 in §2 in connection with Waldspurger's results, cf. [4]. The conjecture when generalized to level > 1 is also true if the corresponding form has weight 2 and is the Yoshida lift of two elliptic cusp forms [6].

iii) Let F be a cuspidal Hecke eigenform and assume that F is in $S_k^*(\Gamma_2)^\perp$ if k is even. Does the function $D_{F,F}(s)$ have any intrinsic arithmetical meaning? (This question was already asked in [33], too; note that $D_{F,F}(s)$ for F as above cannot be proportional to $Z_F(s)$ since $D_{F,F}(s)$ has a pole at $s = k$ while $Z_F(s)$ is holomorphic there, cf. §2). For some numerical computations in this direction in the case $k = 20$ (the first case where $S_k^*(\Gamma_2)^\perp \neq \{0\}$) we refer to [23].

§4. Estimates for Fourier coefficients of Siegel cusp forms

4.1. Results

Very recently, it has turned out that Jacobi forms can be used in a rather simple way to prove growth estimates for Fourier coefficients of Siegel cusp forms of genus 2. The bounds one obtains in this way, in fact, are somewhat better than those obtained previously by different methods.

Let F be a Siegel cusp form of integral weight k on Γ_2 and let $a(T)$ be its Fourier coefficients. The classical Hecke argument immediately gives

$$a(T) \ll_F (\det T)^{k/2} . \tag{7}$$

If one applies a classical theorem of Landau [25, 32] to the Rankin-Dirichlet series

$$\sum_{\{T>0\}/GL_2(\mathbf{Z})} |a(T)|^2 (\det T)^{-s}$$

where the summation extends over a complete set of representatives for the usual left-action of $GL_2(\mathbf{Z})$ on the set of positive definite symmetric half-integral (2, 2)-matrices T, one can sharpen (7) and show that

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - 3/32 + \varepsilon} \qquad (\varepsilon > 0) .$$

(Recall that Landau's theorem roughly speaking asserts that if a Dirichlet series has a meromorphic continuation to $\mathbf{C}$ and satisfies an appropriate functional equation, then one can deduce a "good" upper bound for the growth of its coefficients.) For details we refer to [5] and also [11] where the argument is slightly different; note that the authors prove an estimate for arbitrary genus n.

Let us mention the following

Theorem 1 (Kitaoka [16]). *Suppose that* k *is even. Then*

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - 1/4 + \varepsilon} \qquad (\varepsilon > 0) .$$

The proof of Theorem 1 uses Poincaré series of exponential type on Γ_2 and estimates for generalized matrix-argument Kloosterman sums and can be viewed as a generalization to genus 2 of a well-known method how to obtain "good" bounds for the Fourier coefficients of elliptic cusp forms.

Let us explain now briefly how Jacobi forms can be brought into play (for full details cf. [20, 21]). Let $\phi \in J_{k,m}^{\mathrm{cusp}}$ with Fourier coefficients $c(n, r)$. Then for $k > 2$ one shows that

$$c(n, r) \ll_{\varepsilon, k} (m + |D|^{1/2+\varepsilon})^{1/2} \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} \| \phi \| \quad (\varepsilon > 0) \tag{8}$$

where $D := r^2 - 4mn$ and the bound in $\ll$ only depends on ε and k.

For the proof one carries over the method of Poincaré series and Kloosterman sums from the one-variable situation already mentioned above to the case of the Jacobi group. Note that Poincaré series on Γ_1^J were studied in [14, II, § 2]. The Kloosterman sums that occur in their Fourier coefficients can be related to Salié sums and therefore can easily be estimated (a similar phenomenon happens in the case of modular forms of half-integral weight, cf. [15]). The proof of (8) for D a fundamental discriminant (i.e. the discriminant of a quadratic field) is given in [20, § 1] and for arbitrary D is given in [21, § 1].

On the other hand, if one applies Landau's theorem to the Dirichlet series $D_{F,F}(s)$ discussed in § 3, one finds that

$$\| \phi_m \| \ll_{\varepsilon, F} m^{k/2-2/9+\varepsilon} \quad (\varepsilon > 0) . \tag{9}$$

The estimates (8) and (9) now imply the following

THEOREM 2 [20, 21]. *One has*

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2-13/36+\varepsilon} \quad (\varepsilon > 0) . \tag{10}$$

In fact, both sides of (10) are invariant under $T \mapsto U'TU (U \in GL_2(\mathbf{Z}))$, hence if in (10) we write $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, then we may assume that $m = \min T$, where $\min T$ denotes the least positive integer represented by T. If we use (8) and (9) together with the fact that $\min T \ll (\det T)^{1/2}$ which is well-known from reduction theory, we obtain (10).

4.2. PROBLEMS

i) In [15], Iwaniec using some sophisticated arguments for certain sums of Salié sums showed that the Fourier coefficients $a(n)$ $(n \in \mathbf{N})$ of a cusp form f of weight $k - 1/2$ for $k > 0$ and n squarefree satisfy

$$a(n) \ll_K \sigma_0(n) (\log 2n)^2 n^{k/2-15/28} \| f \| ,$$

where $\sigma_0(n)$ is the number of positive divisors of n and $\| f \|$ is the appropriately normalized Petersson norm of f. We wonder if it is possible to prove an analogous estimate for the Fourier coefficients $c(n, r)$ ($D = r^2 - 4mn$ a fundamental discriminant) of a function $\phi \in J_{k,m}^{\text{cusp}}$ for $k > 2$ which also is analogous to (8) in the sense that an appropriate power of m appears in the denominator on the right-hand side. This then eventually would lead to some improvement of (10) in the case where $-4 \det T$ is a fundamental discriminant.

ii) For arbitrary genus n, the best estimate for Fourier coefficients so far known is due to Böcherer and Raghavan [5] and independently Fomenko [11]. Using Rankin's method they showed that the Fourier coefficients $a(T)$ (T a positive definite symmetric half-integral (n, n)-matrix) of a cusp form F of integral weight k on Γ_n satisfy

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - \delta_n + \varepsilon} \qquad (\varepsilon > 0)$$

where $\delta_n := 2n + 2 + 4\left[\frac{n}{2}\right] + \frac{2}{n+1}$ and $[x]$ = integral part of x (the case $n = 2$ was discussed above).

It is natural to try to apply the method described in 4.1 also for higher genus n. For some results in this direction we refer to [7].

iii) Let F be a non-zero Hecke eigenform in $S_k(\Gamma_2)$ with eigenvalues λ_n and Fourier coefficients $a(T)$.

If k is even and F is in the Maass space $S_k^*(\Gamma_2)$, then

$$\lambda_n \ll_\varepsilon n^{k-1+\varepsilon} \qquad (\varepsilon > 0) \tag{11}$$

and this estimate is best possible as follows from Theorem 2 in §2.

On the other hand, if k is odd or if k is even and $F \in S_k^*(\Gamma_2)^\perp$, then one expects that the generalized Ramanujan-Petersson conjecture holds which predicts that

$$\lambda_n \ll_\varepsilon n^{k-3/2+\varepsilon} \qquad (\varepsilon > 0)\,. \tag{12}$$

To the author's knowledge, the best estimate proved so far for the numbers λ_n is due to Duke, Howe and Li [8] who showed using representation-theoretic methods that

$$\lambda_n \ll_\varepsilon n^{k-1+\varepsilon} \qquad (\varepsilon > 0) \tag{13}$$

provided that n is squarefree (in fact, under the assumption n squarefree the authors proved that $\lambda_n \leqslant \sigma_0(n)^2 n^{k-1}$; it is suggestive that their method, in fact, gives (13) for all n).

In [1, Chap. 2] Andrianov proved that if D is a negative fundamental discriminant and $T_1, \ldots, T_h$ $(h = h(D))$ denotes a set of Γ_1-representatives of positive definite symmetric half-integral (2, 2)-matrices with discriminant D, then

$$\zeta_{\mathbf{Q}(\sqrt{D})}(s - k + 2) \sum_{\nu=1}^{h} \left(\sum_{n \geqslant 1} a(nT_\nu) n^{-s} \right) = \left(\sum_{\nu=1}^{h} a(T_\nu) \right) Z_F(s)\,. \tag{14}$$

Thus – roughly speaking – for fixed T the eigenvalues λ_n are "proportional" to the coefficients $a(nT)$.

Suppose that F is in $S_k^*(\Gamma_2)$. Using Theorem 1 in §2 and the estimate (8) with $m = 1$ one finds that

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - 1/2 + \varepsilon} \quad (\varepsilon > 0) , \tag{15}$$

and (11) together with (14) implies that (15), in fact, is best possible.

On the other hand, taking into account (12) and the fact that the Hecke eigenforms form a basis, one may be led to the following

CONJECTURE 1 [11]. *Let F be a cusp form of integral weight k on Γ_2 and suppose that either k is odd or that k is even and F is in the orthogonal complement of the Maass space. Let $a(T)$ (T a positive definite symmetric half-integral* (2, 2)*-matrix) be the Fourier coefficients of F. Then*

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - 3/4 + \varepsilon} \quad (\varepsilon > 0) .$$

Concerning norms of Fourier-Jacobi coefficients, one optimistically may hope for the truth of the following

CONJECTURE 2. *Let F be a cusp form of integral weight k on Γ_2 and denote by ϕ_m $(m \in \mathbf{N})$ its Fourier-Jacobi coefficients. Then*

$$\| \phi_m \| \ll_{\varepsilon, F} m^{k/2 - 1/2 + \varepsilon} \quad (\varepsilon > 0) . \tag{16}$$

Note that – in view of Theorem 1 of §3 – (17) would be best possible.

NOTE ADDED IN PROOF

The estimates for the Fourier coefficients of cusp forms of arbitrary genus $n \geqslant 2$ obtained in [7] improve upon those obtained in [5, 11], cf. 4.2. ii).

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(Reçu le 22 avril 1992)

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